Topological Susceptibilty in the Trivial Sector with Dynamical Overlap

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Outline

- Introduction
- Topology with Overlap Dirac Operator
- Lattice Setup
- Preliminary Results using JLQCD Dynamical Overlap Configurations with $Q_{top} = 0$
- Conclusion and Outlook

Introduction

Theoretically, topological susceptibility is defined as

$$\chi_{top} = \int d^4x \, \langle \rho(x) \rho(0) \rangle$$

where

$$\rho(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr}[F_{\mu\nu}(x) F_{\lambda\sigma}(x)]$$

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Leutwyler-Smilga relation

$$\chi_{top} = \frac{m_q \Sigma}{n_f} \quad \text{(in the chiral limit)}$$

Introduction (cont)

Since

$$\chi_{top} = \int d^4x \, \langle \rho(x) \rho(0) \rangle = \frac{1}{\Omega} \langle Q_{top}^2 \rangle \,, \quad \Omega = \text{volume}$$

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$$Q_{top} = \int d^4x \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr}[F_{\mu\nu}(x) F_{\lambda\sigma}(x)] = \text{integer}$$

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one can obtain χ_{top} by counting the number of gauge configurations for each topological sector.

However, for a set of gauge configurations in the topologically-trivial sector, $Q_{top} = 0$, it gives $\chi_{top} = 0$

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Thus, one can investigate whether there are topological excitations within any sub-volumes, and to measure the topological susceptibility using the correlation of the topological charges of two sub-volumes.

Introduction (cont)

For any topological sector with Q_{top}, using the translational invariance and the central-limit theorem, one can obtain

$$\lim_{|x-y|\to\infty} \langle \rho(x)\rho(y) \rangle = \frac{Q_{top}^2}{\Omega^2} - \frac{\chi_{top}}{\Omega} + \mathcal{O}(\Omega^{-4})$$

(see T. Onogi's talk)

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Thus, in the trivial sector with $Q_{top} = 0$, for any two widely separated sub-volumes Ω_1 and Ω_2, the correlation of their topological charges would behave as

$$\langle Q_1 Q_2 \rangle \simeq -\frac{\chi_{top}}{\Omega}\Omega_1\Omega_2, \qquad Q_i = \int_{\Omega_i} d^4x \, \rho(x)$$

Introduction (cont)

On a finite lattice, consider two spatial sub-volumes at two time slices t_1 and t_2, measure the time-correlation function

$$C(t_1 - t_2) = \langle Q(t_1) Q(t_2) \rangle = \sum_{\vec{x_1}, \vec{x_2}} \langle \rho(x_1) \rho(x_2) \rangle$$

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However, on a lattice, it is difficult to extract $\rho(x)$ unambiguously from the link variables !

Topology with Overlap Dirac Operator

It is well known that the topological charge density can be defined via the overlap Dirac operator as

$$\rho(x) = \mathrm{tr}[\gamma_5(1 - rD)_{x,x}], \quad r = \frac{1}{2m_0}$$

where D is the overlap Dirac operator

$$D = m_0(1 + V), \quad V = \gamma_5 \frac{H_w}{\sqrt{H_w^2}},$$

$$H_w = \gamma_5(-m_0 + \gamma_\mu t_\mu + W)$$

Topology with Overlap Dirac Operator (cont)

Here $\rho(x) = \mathrm{tr}[\gamma_5(1 - rD)_{x,x}]$ is justified to be a definition of topological charge density since it has been asserted (Kikukawa & Yamada, 1998)

$$\rho(x) \xrightarrow{a \to 0} \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr}[F_{\mu\nu}(x) F_{\lambda\sigma}(x)]$$

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Note that the index theorem on the lattice

$$\text{index}(D) = n_+ - n_- = \sum_x \rho(x) = Q_{top}$$

had been observed by Narayanan and Neuberger in 1995, using the spectral flow of $H_w(m_0)$, before the Ginsparg-Wilson relation was rejuvenated in 1998.

Topology with Overlap Dirac Operator (cont)

It seems natural to use $\rho(x) = \mathrm{tr}[\gamma_5(1 - rD)_{x,x}]$ to compute the topological susceptibility

$$\chi_t = \frac{1}{\Omega}\langle Q_t^2 \rangle = \frac{1}{\Omega}\sum_{x,y}\langle \rho(x)\rho(y) \rangle = \sum_x \langle \rho(x)\rho(0) \rangle$$

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On the other hand, one can derive the relation

$$\text{index}(D) = m \sum_x \mathrm{tr}[\gamma_5(D_c + m)^{-1}_{x,x}] = m \,\mathrm{Tr}[\gamma_5(D_c + m)^{-1}]$$

where

$$D_c = D(1 - rD)^{-1} = 2m_0(1 + V)(1 - V)^{-1}$$

is chirally symmetric but non-local (Chiu & Zenkin, 1998). Note that for the topologically-trivial configurations , D_c is well-defined (without any poles).

Topology with Overlap Dirac Operator (cont)

Thus one can regard

$$\rho_1(x) = m \, \mathrm{tr}[\gamma_5 (D_c + m)^{-1}_{x,x}]$$

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Obviously, the identity index$(D) = m \, \mathrm{Tr}[\gamma_5 (D_c + m)^{-1}]$ can be generalized to

$$\text{index}(D) = m_1 m_2 \cdots m_k \mathrm{Tr}[\gamma_5 (D_c + m_1)^{-1} (D_c + m_2)^{-1} \cdots (D_c + m_k)^{-1}]$$

with the generalized topological charge density

$$\rho_k(x) = m_1 m_2 \cdots m_k \mathrm{tr}[\gamma_5 (D_c + m_1)^{-1} (D_c + m_2)^{-1} \cdots (D_c + m_k)^{-1}]_{x,x}$$

Topology with Overlap Dirac Operator (cont)

Presumably, any ρ_k can be used to compute χ_{top}.

In general,

$$\chi_{top} = \frac{m_1 \cdots m_k m_{k+1} \cdots m_l}{\Omega} \langle \mathrm{Tr}[\gamma_5 (D_c + m_1)^{-1} \cdots (D_c + m_k)^{-1}] \times \mathrm{Tr}[\gamma_5 (D_c + m_{k+1})^{-1} \cdots (D_c + m_l)^{-1}] \rangle$$

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It has been pointed out by Lüscher, for $k \geq 2$ and $l \geq 5$, χ_{top} avoids the short-distance singularities in the continuum limit.

Re-derive the Leutwyler-Smilga relation (Chandrasekharan,'98)

Consider the ($p = 0$) flavor-singlet pseudoscalar (η') correlator

$$
\begin{aligned}
G_{\eta'} &= \frac{1}{\Omega} \sum_{x,y} \langle \bar\psi(x)\gamma_5\psi(x)\bar\psi(y)\gamma_5\psi(y) \rangle \\
&= \frac{1}{\Omega Z} \int [dU] \, \mathrm{det}(D) e^{-A_g[U]} \times \\
&\quad \left\{ \mathrm{Tr}[(D_c + m_q)^{-1}\gamma_5 (D_c + m_q)^{-1}\gamma_5] - \left(\mathrm{Tr}[(D_c + m_q)^{-1}\gamma_5] \right)^2 \right\} \\
&= \frac{1}{\Omega Z} \int [dU] \, \mathrm{det}(D) e^{-A_g[U]} \left\{ \frac{1}{m_q} \mathrm{Tr}(D_c + m_q)^{-1} - \left[\frac{n_f}{m_q} (n_+ - n_-) \right]^2 \right\}
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In the chiral limit $m_q \to 0$, if η' stays massive, then its propagator $G_{\eta'} \propto m_{\eta'}^{-2}$ must be non-singular, which in turn implies

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\chi_{top} = \frac{1}{\Omega} \langle (n_+ - n_-)^2 \rangle = \frac{m_q}{n_f} \Sigma, \qquad \Sigma \equiv \frac{1}{n_f \Omega} \mathrm{Tr}(D_c + m_q)^{-1}
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Lattice Setup (See Talks by Hashimoto, Kaneko, and Onogi)

- Lattice size: $16^3 \times 32$
- Gluons: Iwasaki gauge action at $\beta = 2.30$
- Quarks ($n_f = 2$): overlap Dirac operator with $m_0 = 1.6$
- Add extra Wilson fermions and pseudofermions

$$\det(H_{ov}^2) \longrightarrow \det(H_{ov}^2)\frac{\det(H_w^2)}{\det(H_w^2+\mu^2)}, \quad \mu = 0.2$$

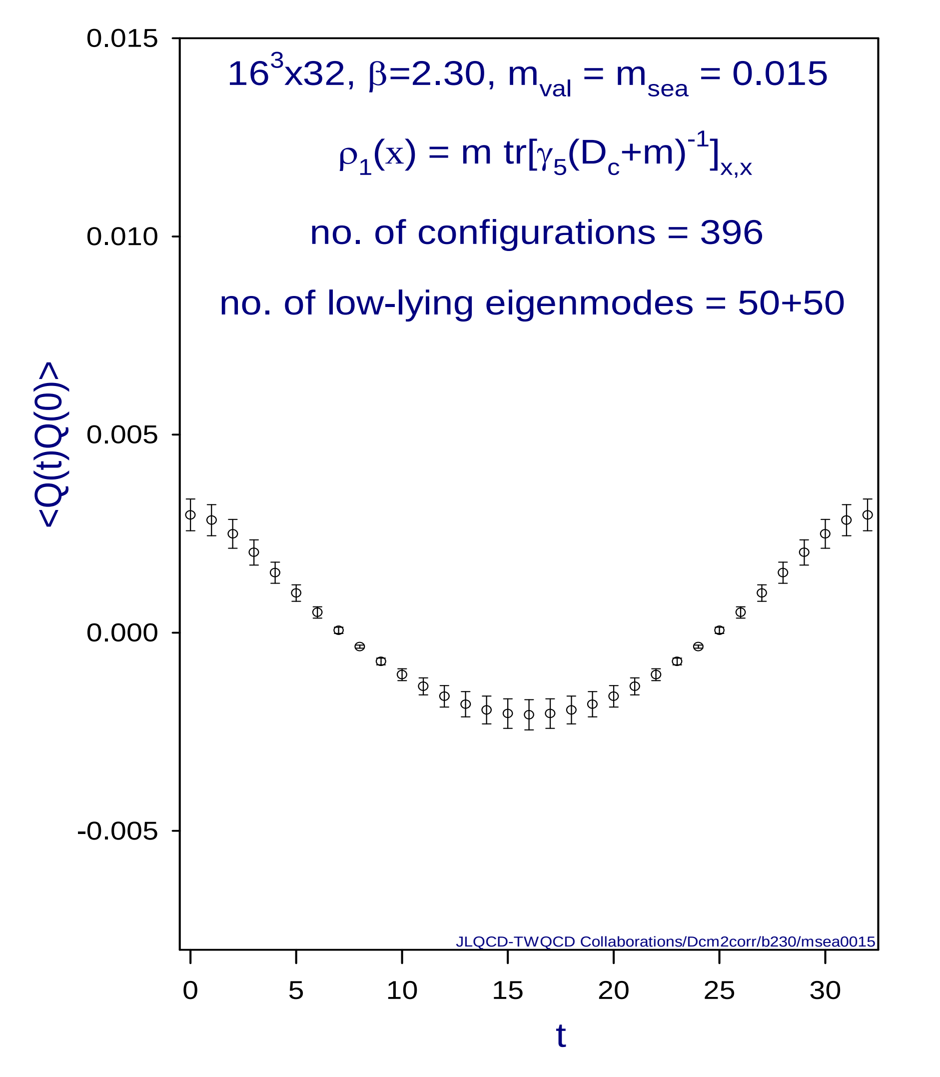
to forbid $\lambda(H_w)$ crossing zero, thus Q_{top} is invariant.

- Quark masses: m_{sea} = 0.015, 0.025, 0.035, 0.050, 0.070, 0.100, each of ~ 1000 confs with $Q_{top} = 0$.
- For each configuration, 50+50 low-lying eigenmodes of overlap Dirac operator are projected.

Preliminary Results (using 396 confs for each m_{sea})

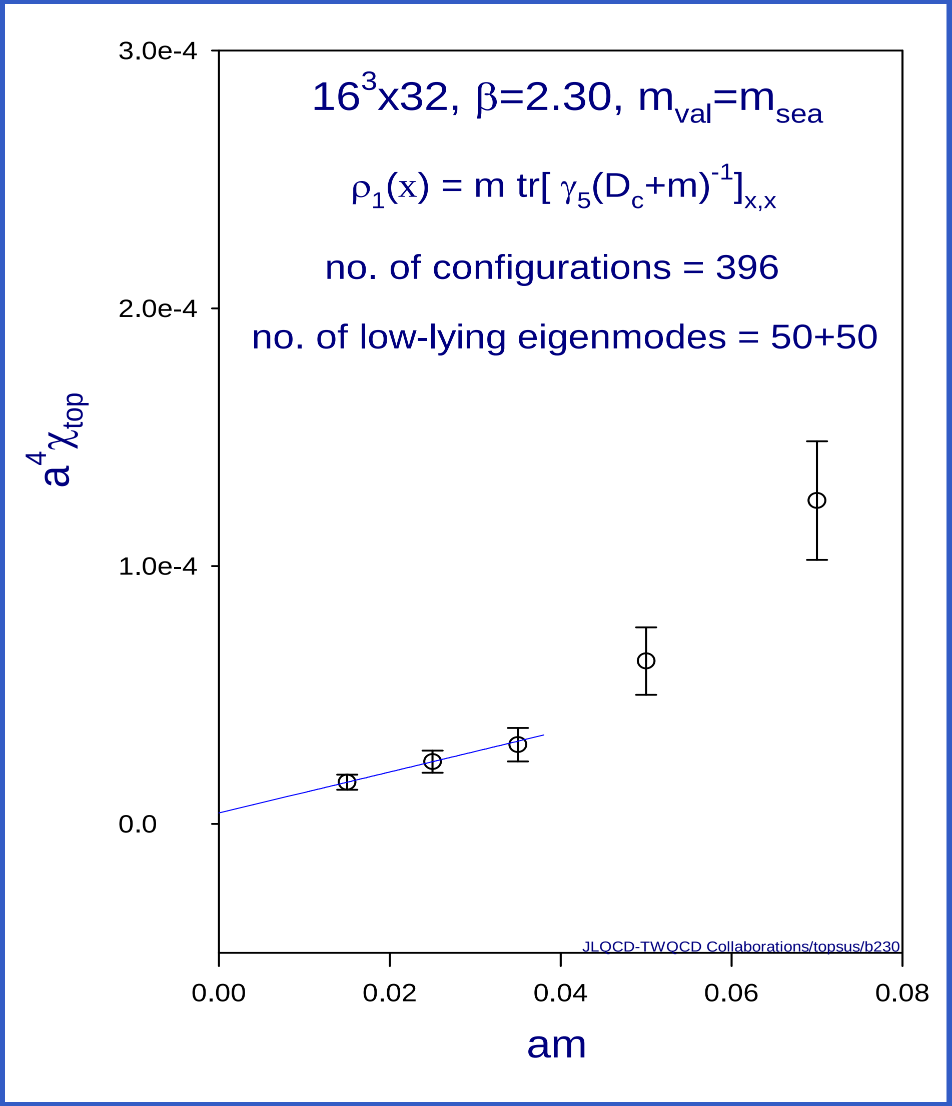
On the $16^3 \times 32$ lattice, measure the time-correlation function

$$C(t_1 - t_2) = \langle Q(t_1) Q(t_2) \rangle = \sum_{\vec{x_1}, \vec{x_2}} \langle \rho(x_1) \rho(x_2) \rangle$$

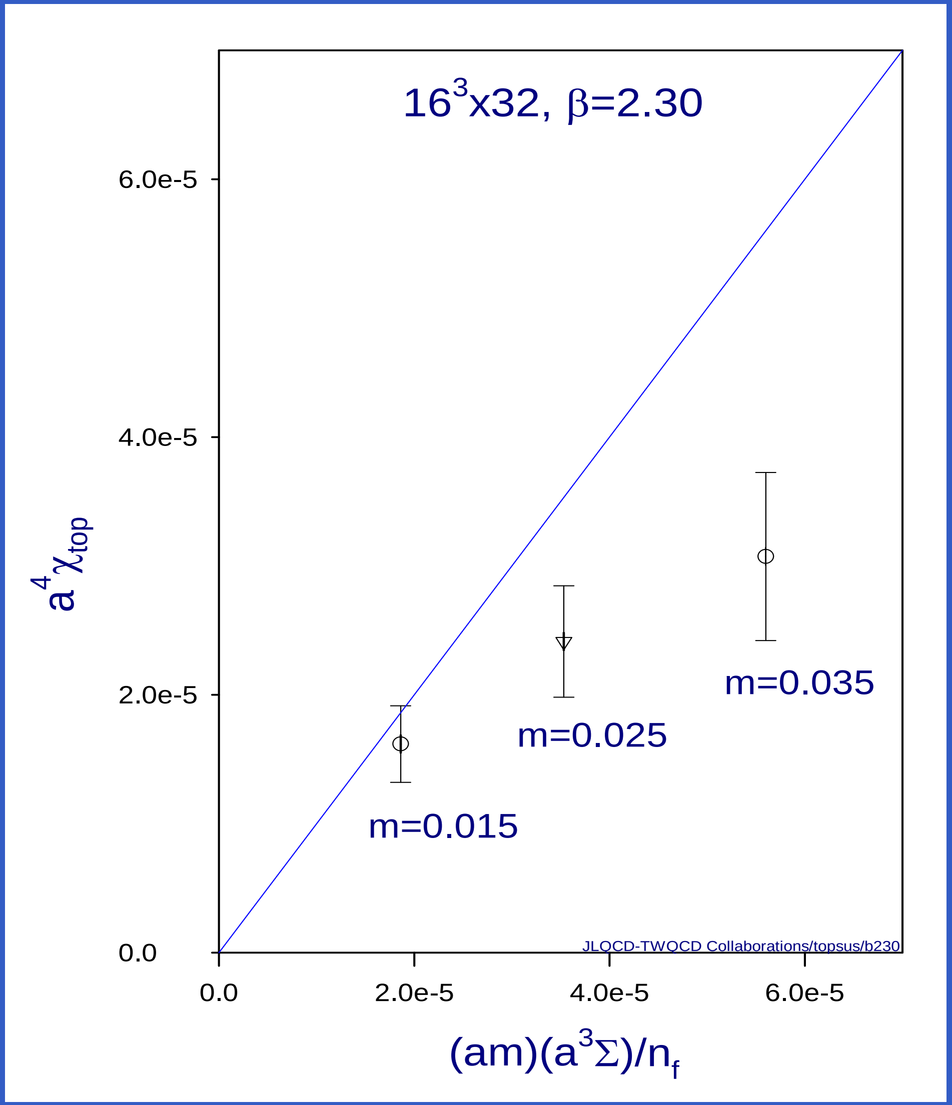


Topological Susceptibility

$$a^4 \chi_{top} = -\frac{32}{16^3} \langle Q(t_1) Q(t_2) \rangle , \qquad |t_1 - t_2| = 16$$

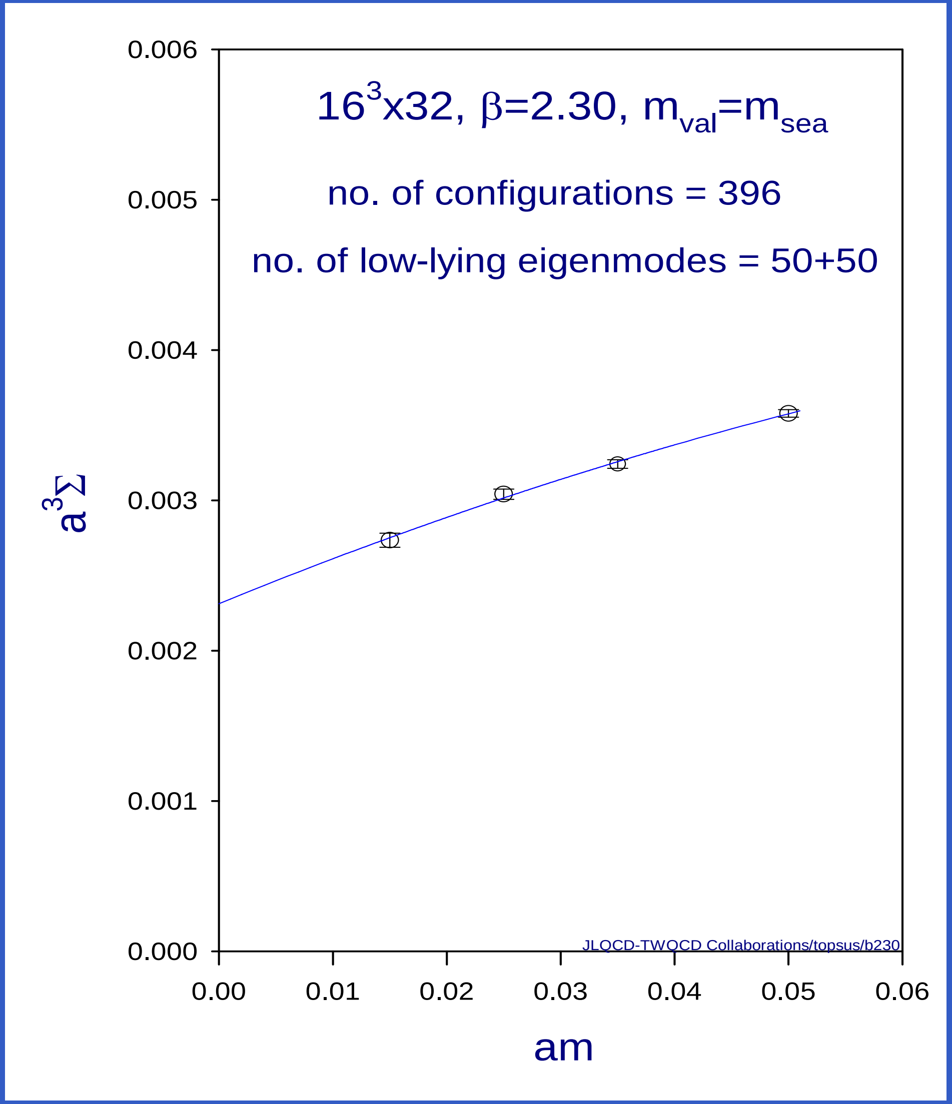


Realization of Leutwyler-Smilga relation



In the limit $m \to 0$, $\chi_{top} \to m\Sigma/n_f$, in agreement with the Leutwyler-Smilga relation !

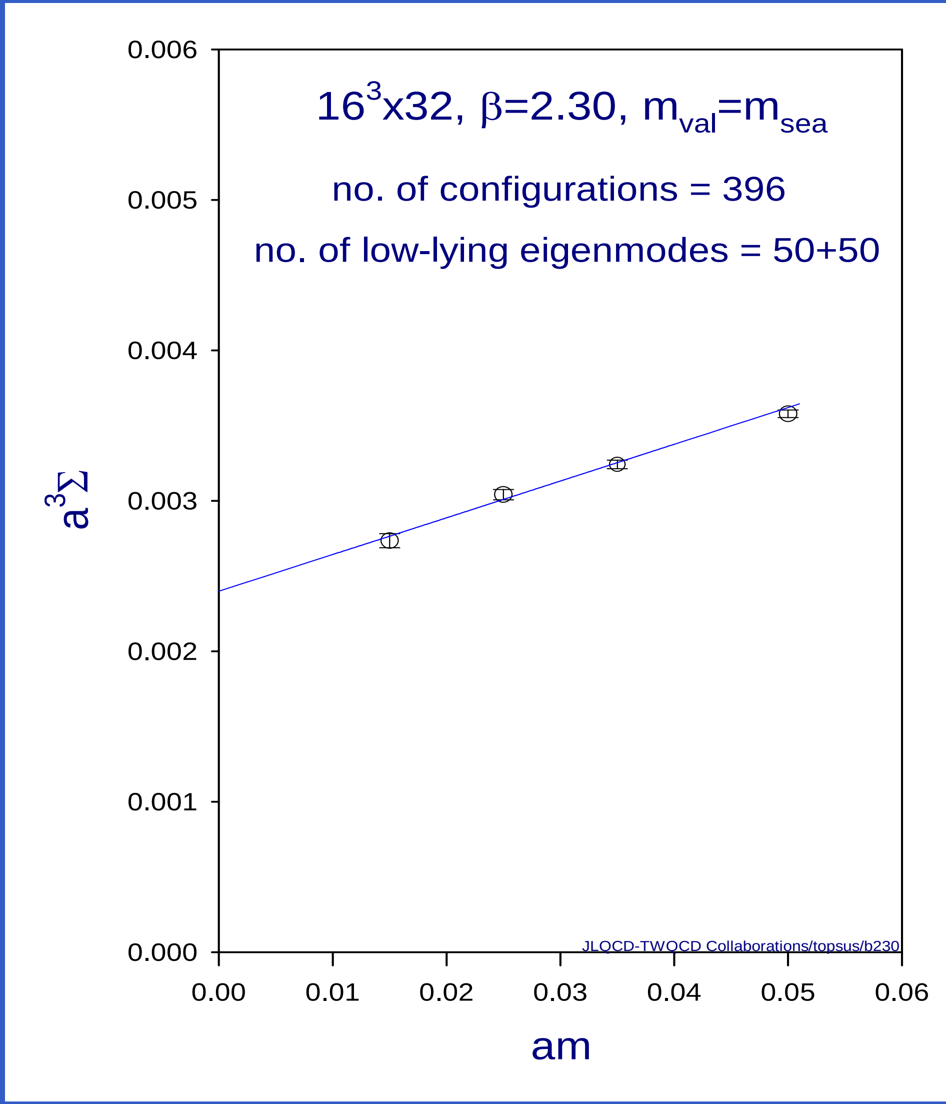
Chiral Condensate



$$\lim_{m\to 0} a^3\Sigma = 0.0023(1) \quad \text{(quadratic fit)}$$

Use $a^{-1} = 1690$ MeV, $Z_s = 1.14(2)$, $\Sigma^{\overline{MS}}(2\text{ GeV}) = (233 \pm 5\text{ MeV})^3$

Chiral Condensate (cont)



$$\lim_{m\to 0} a^3\Sigma = 0.0024(1) \quad \text{(linear fit)}$$

Use $a^{-1} = 1690$ MeV, $Z_s = 1.14(2)$, $\Sigma^{\overline{MS}}(2\ \text{GeV}) = (236 \pm 5\ \text{MeV})^3$

Conclusion and Outlook

- For the topologically-trivial gauge configurations generated with $n_f = 2$ dynamical overlap quarks constrainted by extra Wilson and pseudofermions, they possess topologically non-trivial excitations (e.g., instanton and anti-instanton pairs) in sub-volumes.

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- These near-zero modes allow us to determine χ_{top} and Σ.
- In the chiral limit, the Leutwyler-Smilga relation is realized !
- Similar studies for $Q_{top} = 2$, and $Q_{top} = 4$ sectors are now in progress.