



# The Kaon Bag Parameter

## *from 2+1-Flavors of Domain-Wall Fermion*

Saul D. Cohen

`sdcohen@phys.columbia.edu`

RBC and UKQCD Collaborations

(Columbia University)

Calculations done using the QCDOC supercomputers  
at RIKEN-BNL Research Center and the University of Edinburgh





# Outline

## The Kaon Bag Parameter

The bag parameter is needed to compare  $K$  and  $B$  physics in the unitarity triangle. The lattice provides the only first-principles method for determining  $B_K$ .

## The Challenges

- Realistic sea content with large volume
- Reliable renormalization
- Controlled operator mixing
- Extrapolation to continuum limit

## New Result

We now have a lattice  $B_K$  measurement for 2+1 flavors, using domain-wall fermions, providing good chiral symmetry and nonperturbative renormalization.

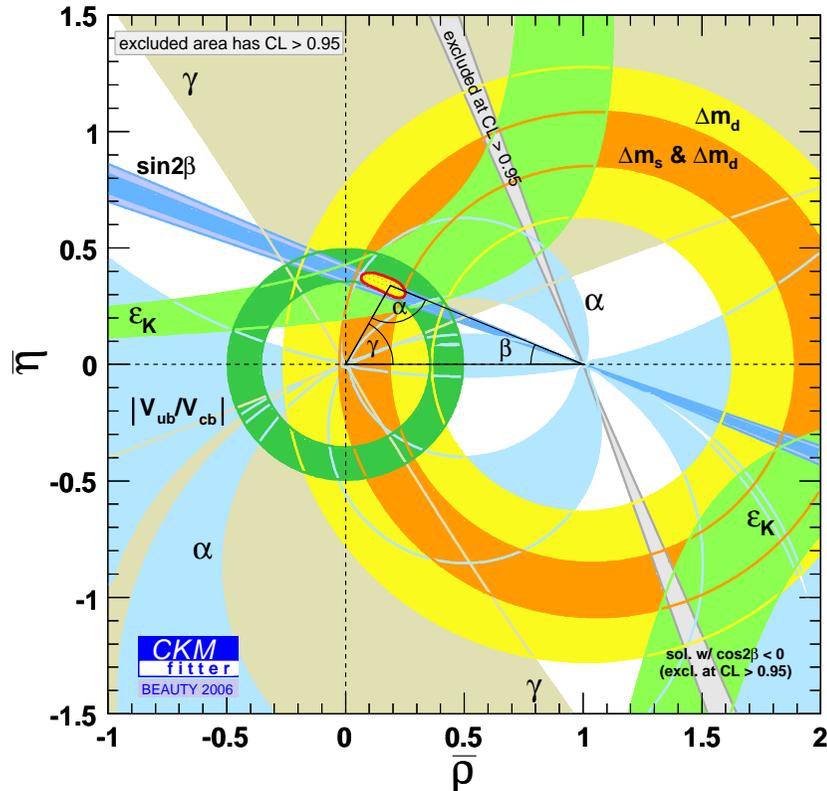




# The Kaon Bag Parameter

## Important to CP Violation

Errors in the determination of  $\epsilon$  are now dominated by uncertainty in the value of  $B_K$ .



$$\epsilon = \hat{B}_K \text{Im}\lambda_t \frac{G_F^2 f_K^2 M_K M_W^2}{6\sqrt{2}\pi^2 \Delta M_K} \times \{ \text{Re}\lambda_c [\eta_1 S_0(x_c) - \eta_3 S_0(x_c, x_t)] - \text{Re}\lambda_t \eta_2 S_0(x_t) \} e^{i\pi/4}$$

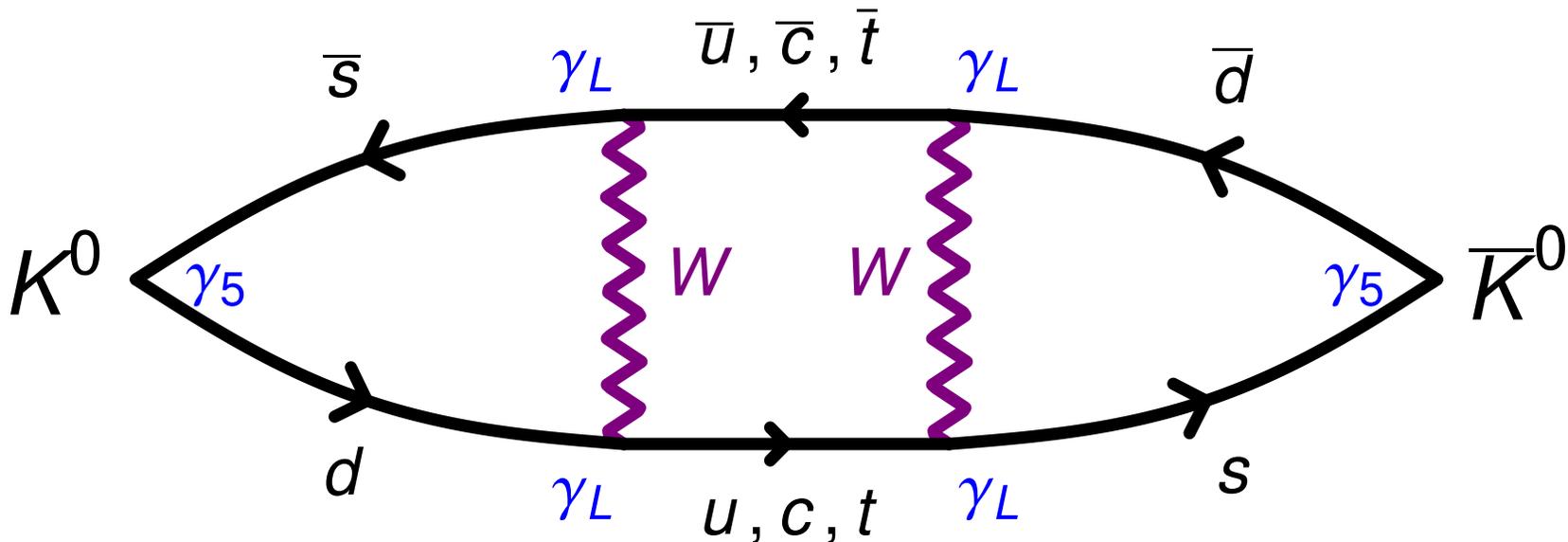
Constraint by  $\epsilon$  denoted by green hyperbolic bands.



# The Kaon Bag Parameter

## Fundamental Diagram

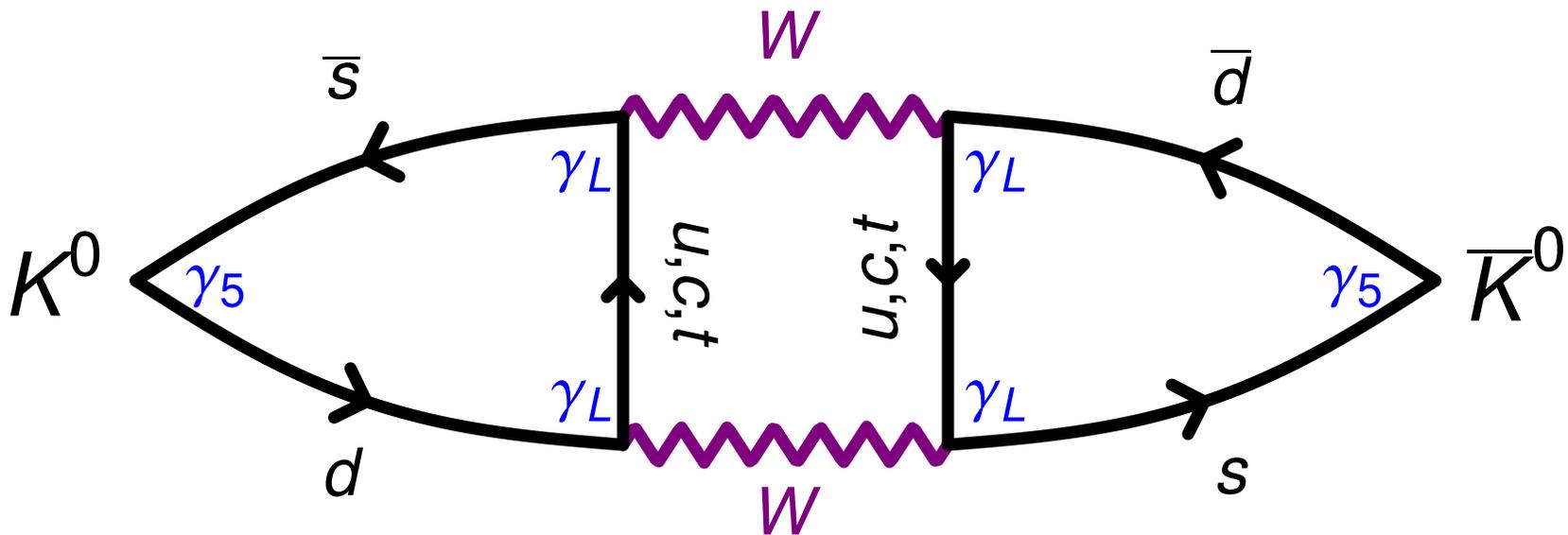
$B_K$  parametrizes the amount of mixing between neutral kaons due to weak interactions. At leading order in the Standard Model, we understand this mixing is due to the exchange of two  $W$  bosons:



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## Fundamental Diagram

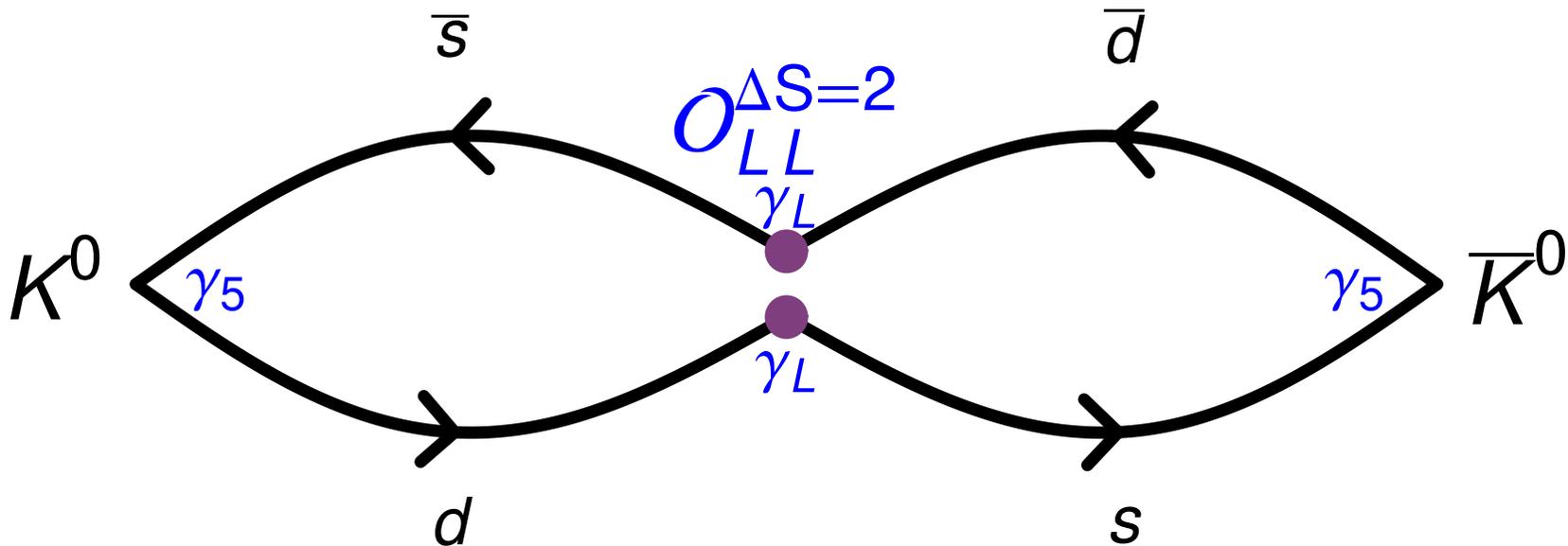
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# The Kaon Bag Parameter

## Figure-Eight Diagram

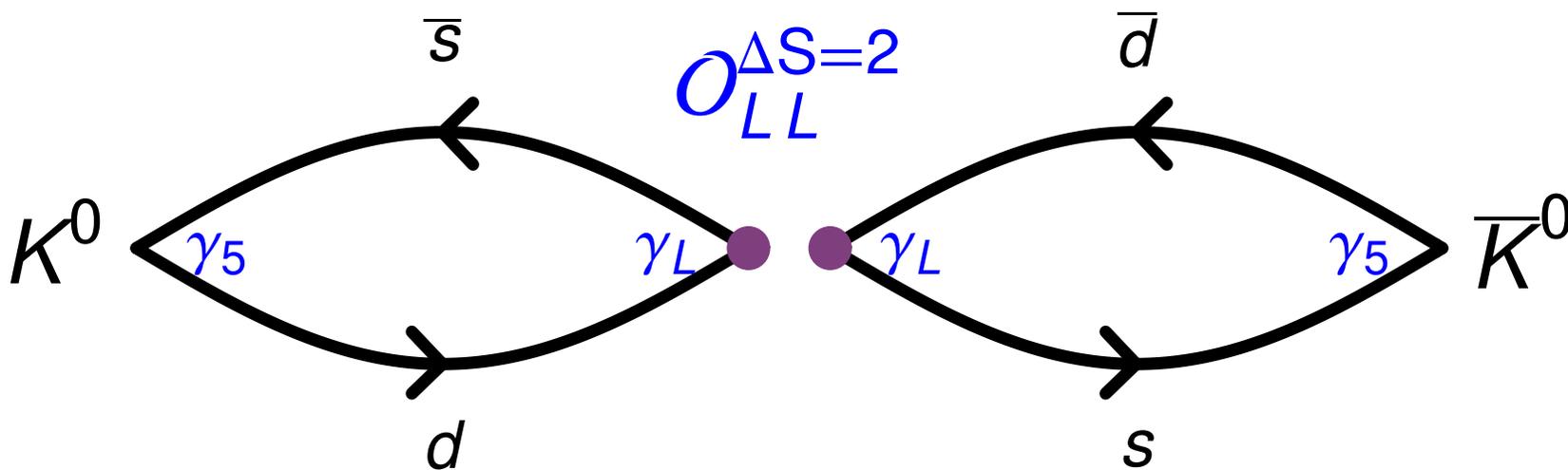
After integrating out the heavy quarks and weak bosons, we evaluate this QCD diagram. Operators with quantum numbers of the kaon are inserted at the initial and final times, creating a kaon and annihilating an anti-kaon respectively.



# The Kaon Bag Parameter

## Figure-Eight Diagram

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# The Kaon Bag Parameter

## Definition

The operator inserted at the center is the four-quark left-left operator that changes strangeness by two:

$$B_K = \frac{\langle \overline{K^0} | \mathcal{O}_{LL}^{\Delta S=2} | K^0 \rangle}{\frac{8}{3} f_K^2 M_K^2}$$

$$\mathcal{O}_{LL}^{\Delta S=2} = (\bar{s}d)_L (\bar{s}d)_L$$

$$\mathcal{O}(\mu) = \sum_i Z_i(\mu) \mathcal{O}_i^{\Delta S=2}$$



# The Challenge

## Operator Mixing

In the continuum,  $B_K$  contains only the operator of the form  $VV + AA$ , which renormalizes multiplicatively:

$$\mathcal{O}_{VV+AA}^{\Delta S=2} = (\bar{s}\gamma_\mu d)(\bar{s}\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}\gamma_\mu\gamma_5 d)$$

but if chiral symmetry is broken on the lattice, operators with other chiral structures may mix under renormalization. These operators diverge relative to  $\mathcal{O}_{LL}$  in the chiral limit.

$$\mathcal{O}_{VV-AA}^{\Delta S=2} = (\bar{s}\gamma_\mu d)(\bar{s}\gamma_\mu d) - (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}\gamma_\mu\gamma_5 d)$$

$$\mathcal{O}_{SS\pm PP}^{\Delta S=2} = (\bar{s}d)(\bar{s}d) \pm (\bar{s}\gamma_5 d)(\bar{s}\gamma_5 d)$$

$$\mathcal{O}_{TT}^{\Delta S=2} = (\bar{s}\sigma_{\mu\nu} d)(\bar{s}\sigma_{\mu\nu} d)$$





# The Solution

## Operator Mixing

In the continuum,  $B_K$  contains only the operator of the form  $VV + AA$ , which renormalizes multiplicatively:

$$\mathcal{O}_{VV+AA}^{\Delta S=2} = (\bar{s}\gamma_\mu d)(\bar{s}\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}\gamma_\mu\gamma_5 d)$$

## Domain-Wall Fermions

The domain-wall formalism allows us to control chiral symmetry breaking; as long as our residual masses small, we avoid the unwanted terms which contribute at  $O(m_{\text{res}}^2)$ .

In this framework, we may also apply continuum partially quenched chiral perturbation theory and use non-perturbative renormalization.





# RBC/UKQCD Lattices

## Ensemble Status

Generation of the smaller  $16^3$  volume run is now completed, and work based on these lattices is already posted to arXiv (hep-lat/0701013, hep-ph/0702042). Analysis is in progress for the larger  $24^3$  volume.

$\beta$	2.13	$c_1$	-0.331	$16^3 \times 32$	$24^3 \times 64$
$L_s$	16			every 20 <sup>th</sup>	every 80 <sup>th</sup>
$am_s$	0.04	$am_l$	0.005	—	(43 confs)
			0.010	(150 confs)	(31 confs)
			0.020	(150 confs)	—
			0.030	(150 confs)	—



# RBC/UKQCD Lattices



## Lattice Scale

The lattice scale may be computed from the static quark potential, vector meson ( $\rho$ ) mass, or the method of planes. All agree:

$$a^{-1} = 1.62(4) \text{ GeV}, \quad a = 0.122(3) \text{ fm}$$
$$V \approx (2 \text{ fm})^3$$

## Residual Mass

The residual mass is determined from the midpoint correlator, and is also the (negative) quark mass at which the pseudoscalar mass extrapolates to zero.

$$am_{\text{res}} = 0.00308(4)$$



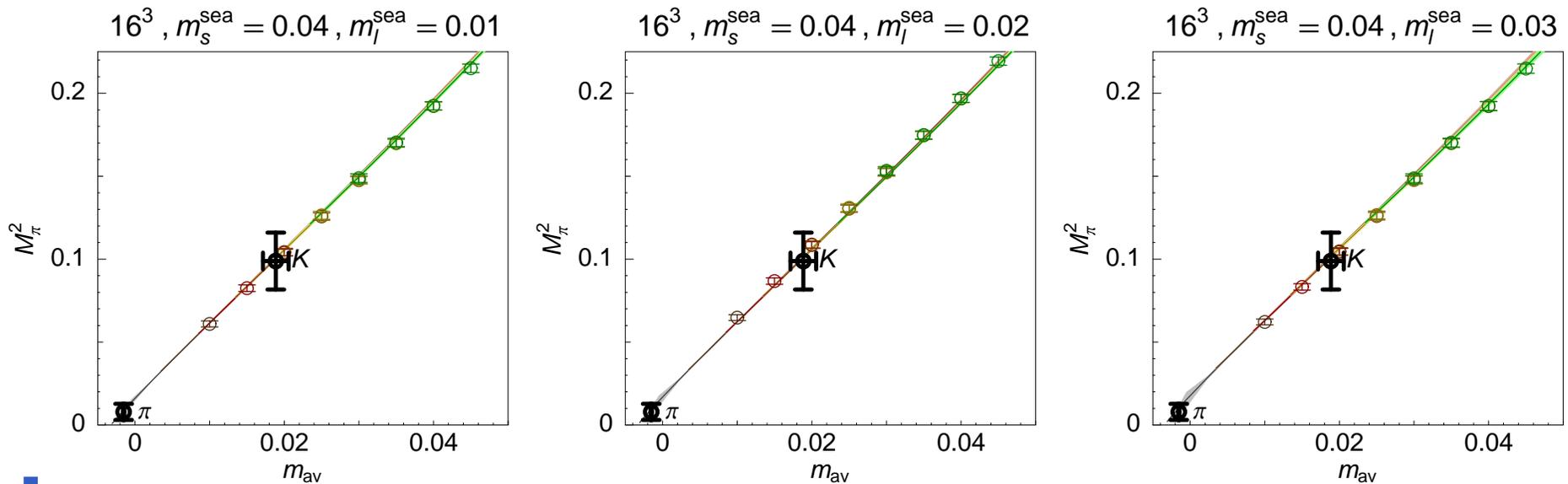


# $16^3 \times 32$ Results

## Quark Masses

We set the bare light and strange quark masses by fixing the pseudoscalar masses to the physical pion and kaon masses. We find  $am_l = 0.00162(8)$ ,  $am_s = 0.0390(21)$ . We use the lowest-order fit from chiral perturbation theory:  $M_P^2 = 2\mu(m_{av} + m_{res})$ .

$M_P^2$  2+1f Partially Quenched Chiral Fit





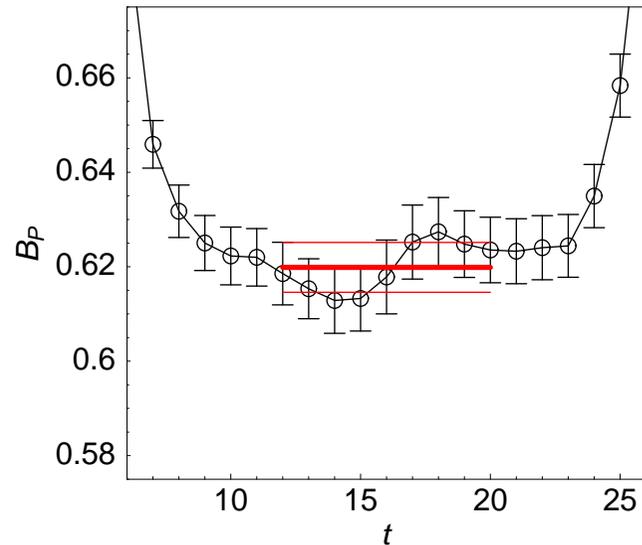
# $16^3 \times 32$ Results

## Three-Point Plateaux

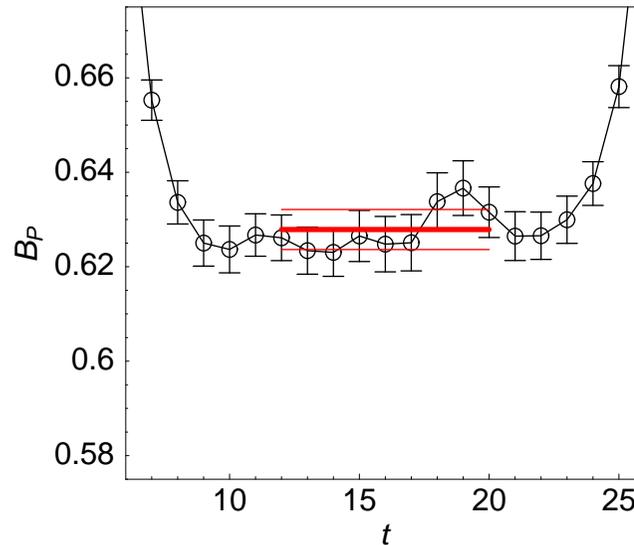
We expect the matrix element to approach its asymptotic value far from the source and sink. Depicted here are the unitary quark masses (where  $m_{\text{val}} = m_{\text{sea}}$ ).

$B_P$  Plateaux

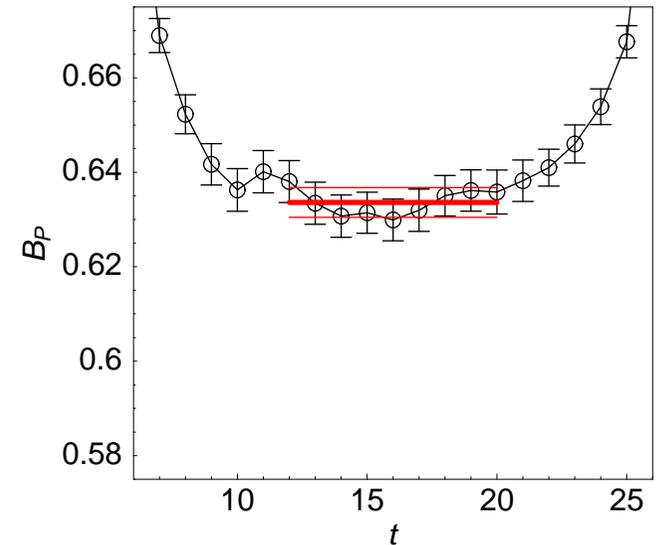
$16^3, m_s^{\text{sea}} = 0.04, m_l^{\text{sea}} = 0.01$



$16^3, m_s^{\text{sea}} = 0.04, m_l^{\text{sea}} = 0.02$



$16^3, m_s^{\text{sea}} = 0.04, m_l^{\text{sea}} = 0.03$



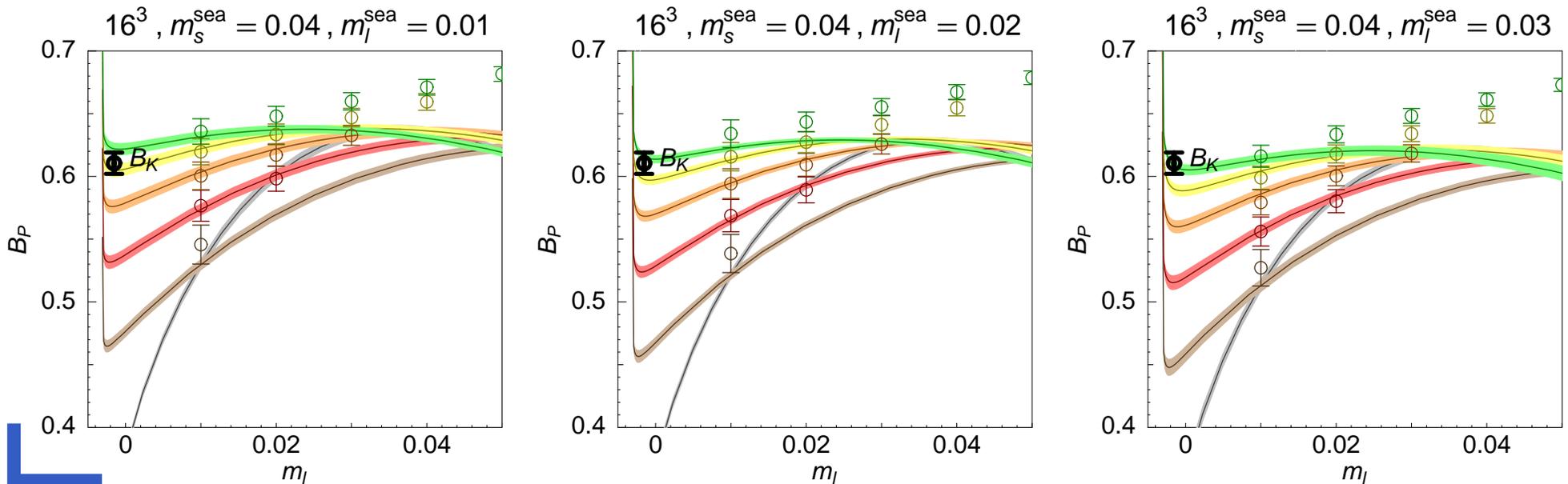


# $16^3 \times 32$ Results

## Kaon Bag Parameter

$B_K$  is computed for 15 nondegenerate combinations of valence strange and light masses. Light mass runs along the  $x$ -axis; strange mass is denoted by color. We fit to the partially quenched 2+1 flavor NLO chiral perturbation theory form of Sharpe and Van de Water.  $B_K^{\text{bare}} = 0.607(9)$

$B_P$  2+1f Partially Quenched Chiral Fit



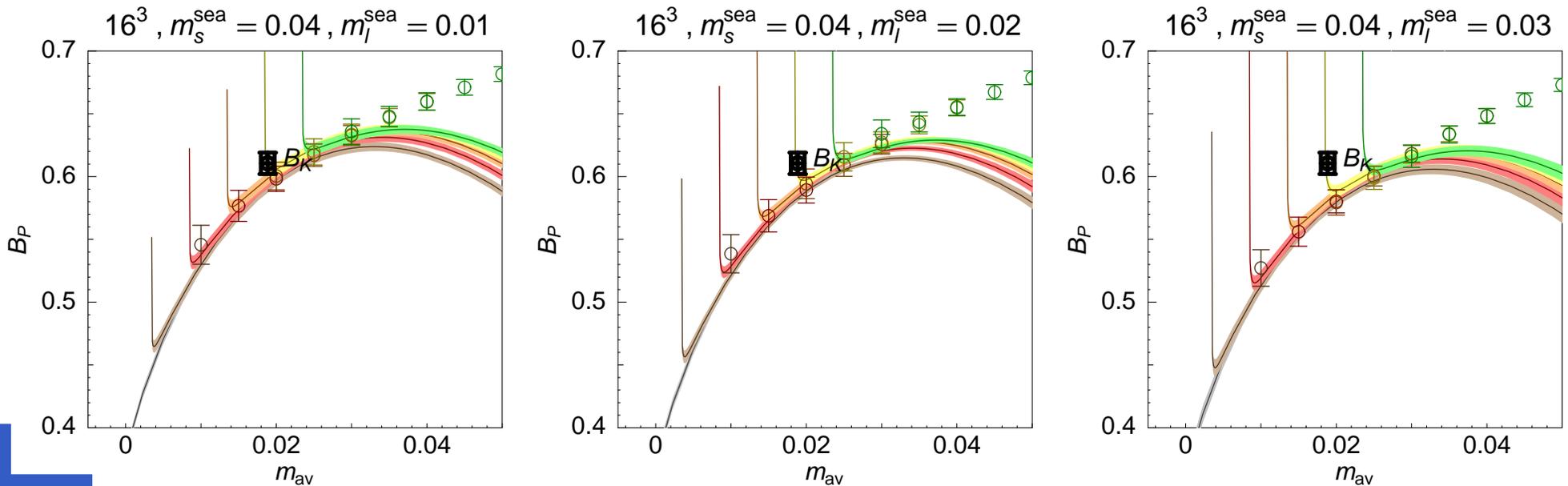


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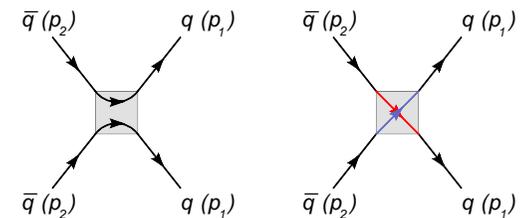
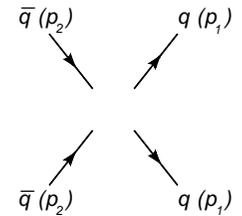
$B_P$  2+1f Partially Quenched Chiral Fit



# Nonperturbative Renormalization

## Method

1. Calculate Fourier transform of point-source propagators.
2. Combine the propagators according to the structure of the matrix element.
3. Amputate the external legs.
4. After projection, this matching (including quark field renormalization  $Z_q$ ) gives the renormalization.



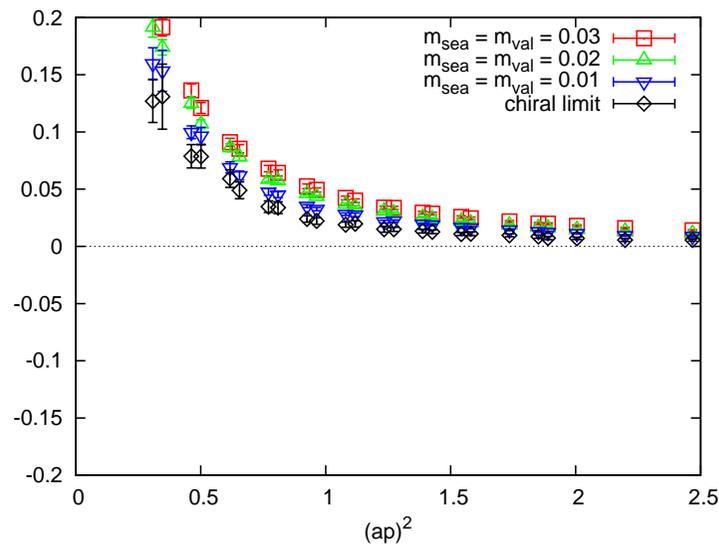
$$Z_q^{-2} \hat{Z} \left[ \text{Diagram with box and external legs} \right] = \left[ \text{Diagram with box and external legs} \right] \otimes \left[ \text{Diagram with box and external legs} \right]$$

# Nonperturbative Renormalization

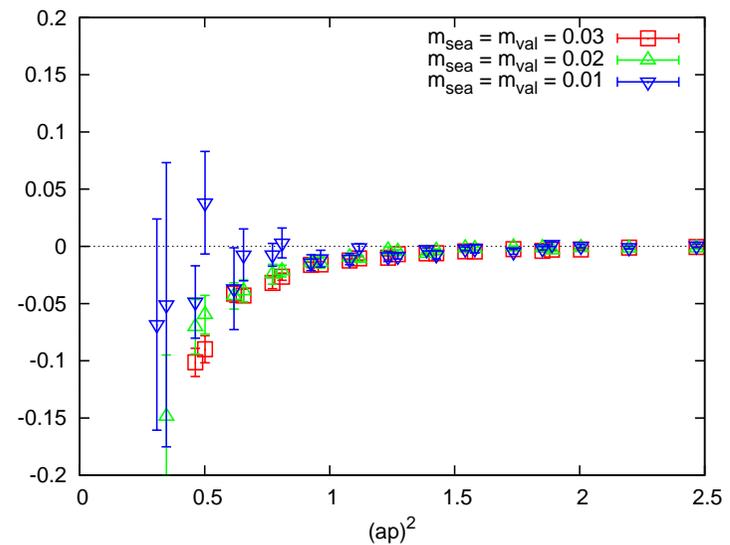
## Operator Mixing

First, we verify that the undesired mixings are indeed small.

$$Z_{VV+AA, VV-AA}$$



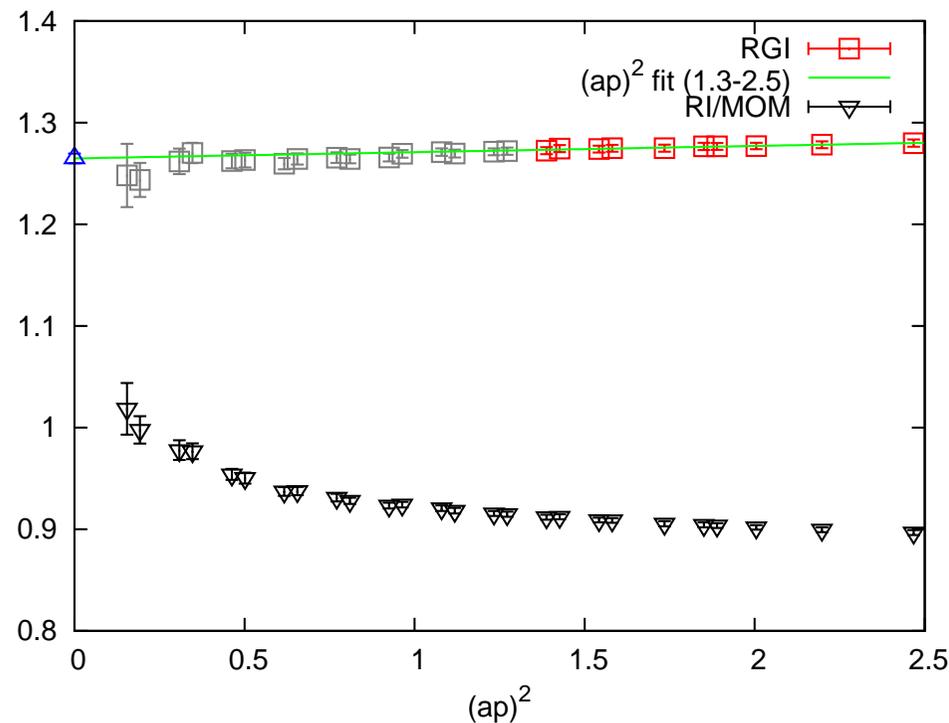
$$Z_{VV+AA, SS-PP}$$



# Nonperturbative Renormalization

## Lattice-to-Continuum Matching

$Z_{VV+AA, VV+AA}$  can be divided into a constant  $Z^{\text{RGI}}$  and a running  $f(\mu)$ .

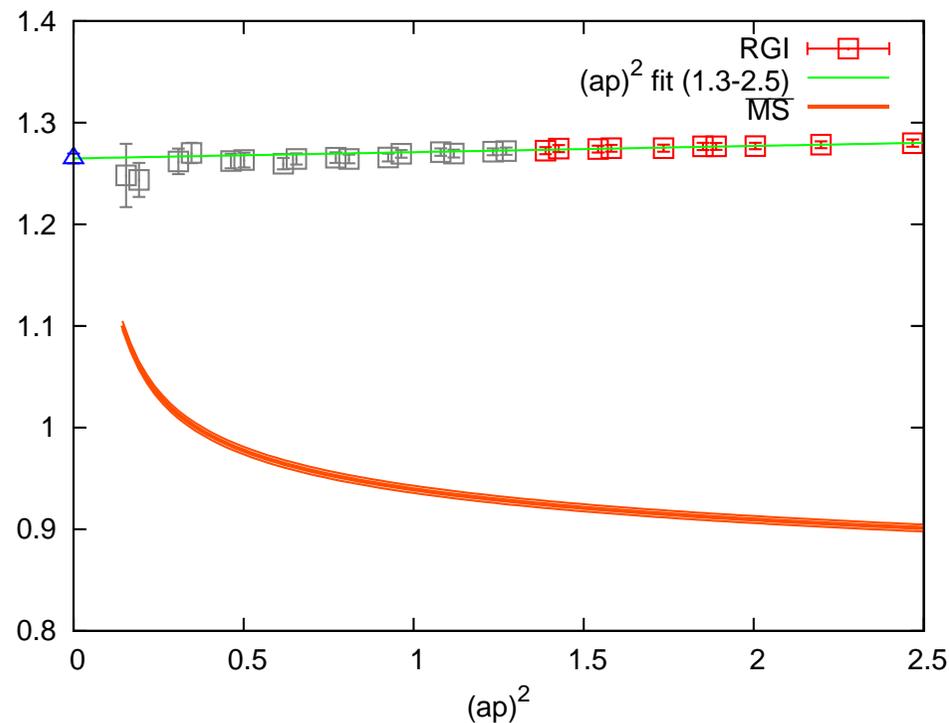


# Nonperturbative Renormalization

## Lattice-to-Continuum Matching

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We may simply apply perturbation theory to find  $Z_{B_K}^{\overline{\text{MS}}}(2 \text{ GeV})$ .

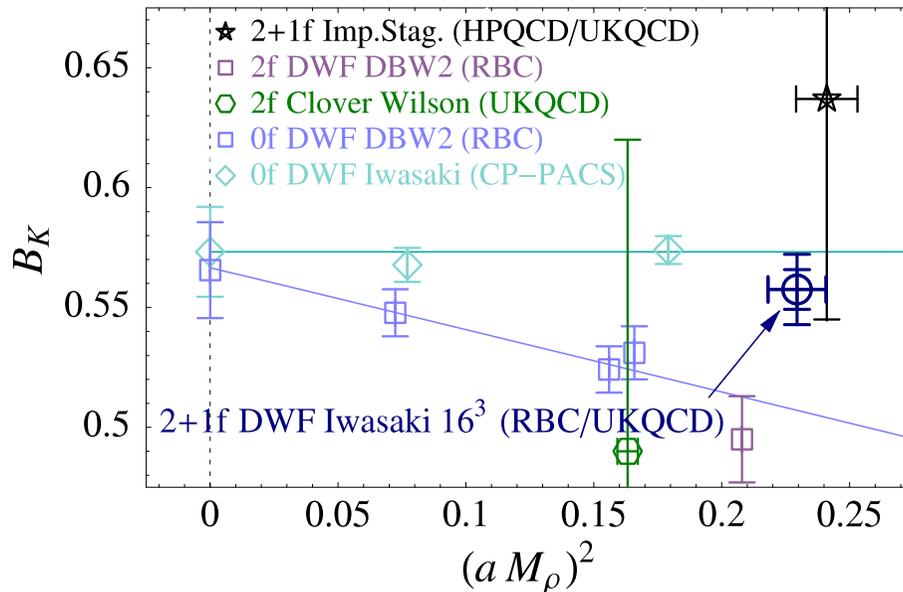




# $16^3 \times 32$ Results

## Comparisons

Here we compare our 2+1 flavor domain-wall fermion  $B_K$  result with other chiral fermion and unquenched results.



With  $Z_{B_K} = 0.917(07)(18)$   
in  $\overline{\text{MS}}$  at 2 GeV

$$B_K = 0.557(12)(16)$$

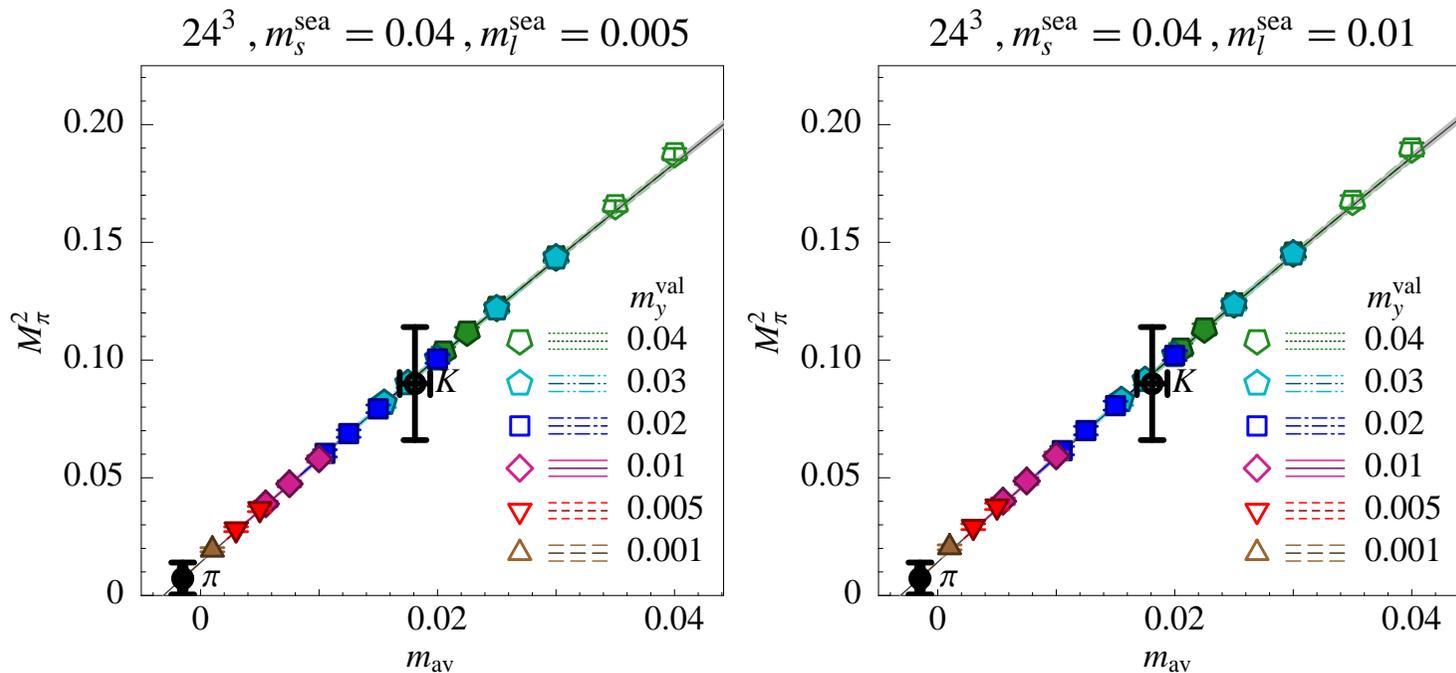


# $24^3 \times 64$ Results

## Quark Masses

Analysis is in progress on a larger volume  $(3 \text{ fm})^3$  at the same scale, using even smaller sea and valence masses: down to  $am_{\text{val}} = 0.005$  and  $am_{\text{val}} = 0.001$ .

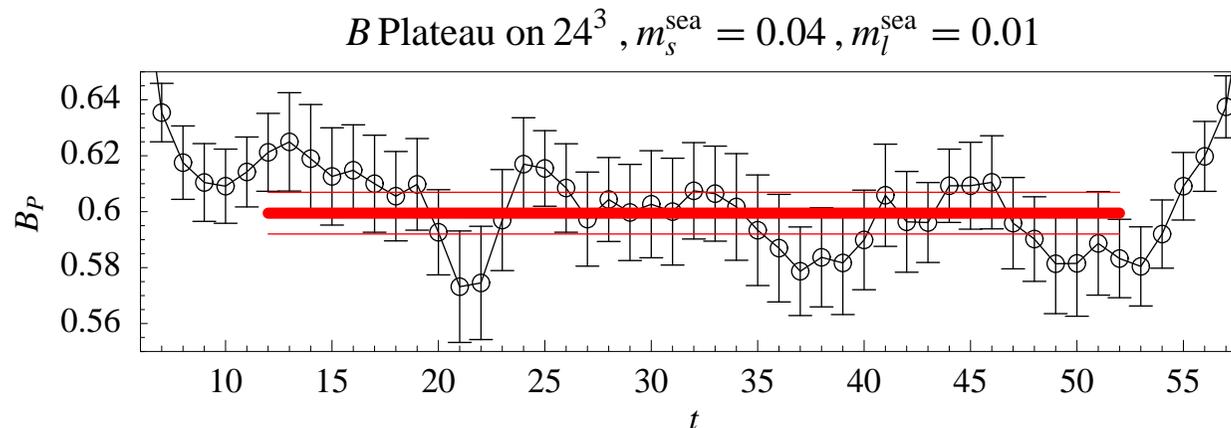
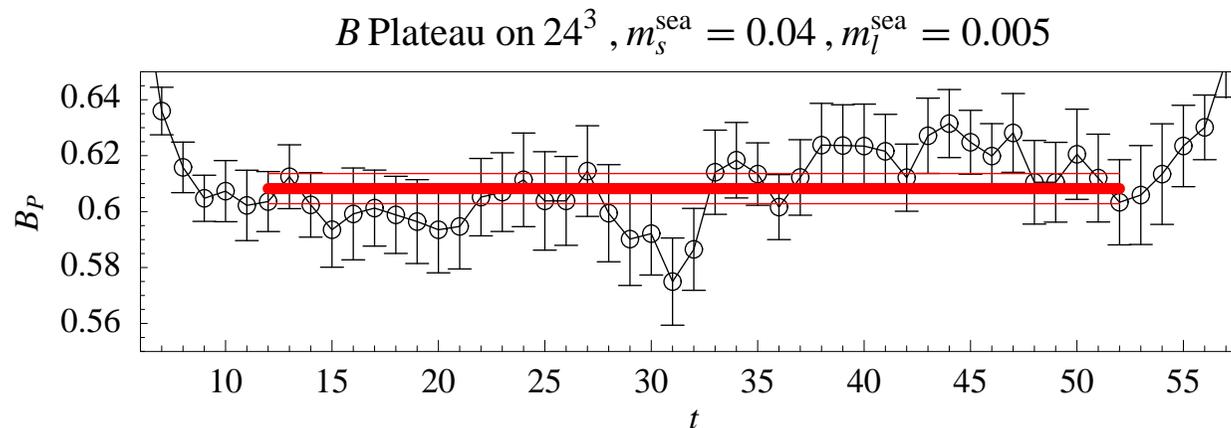
$M_p^2$  2+1f Partially Quenched Chiral Fit



# $24^3 \times 64$ Results

## Three-Point Plateaux

The much longer length of the time-dimension allows us to average over more fluctuations in the gauge field, yielding smaller statistical errors.

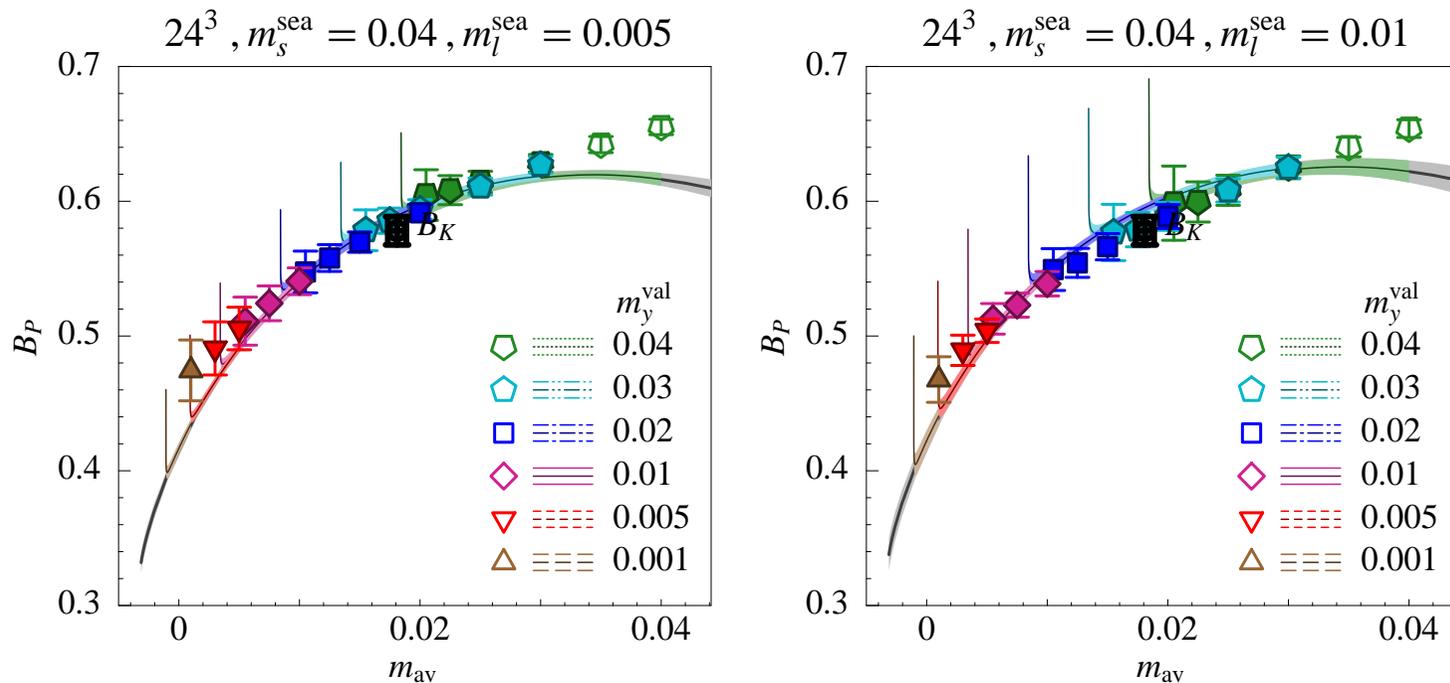


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## Kaon Bag Parameter

Already, on only 31 and 43 configurations, we nearly match the statistical error of the  $16^3$  ensemble. Large-volume data will allow us to test the chiral form at very low mass and take into account finite-volume effects.

$B_P$  2+1f Partially Quenched Chiral Fit

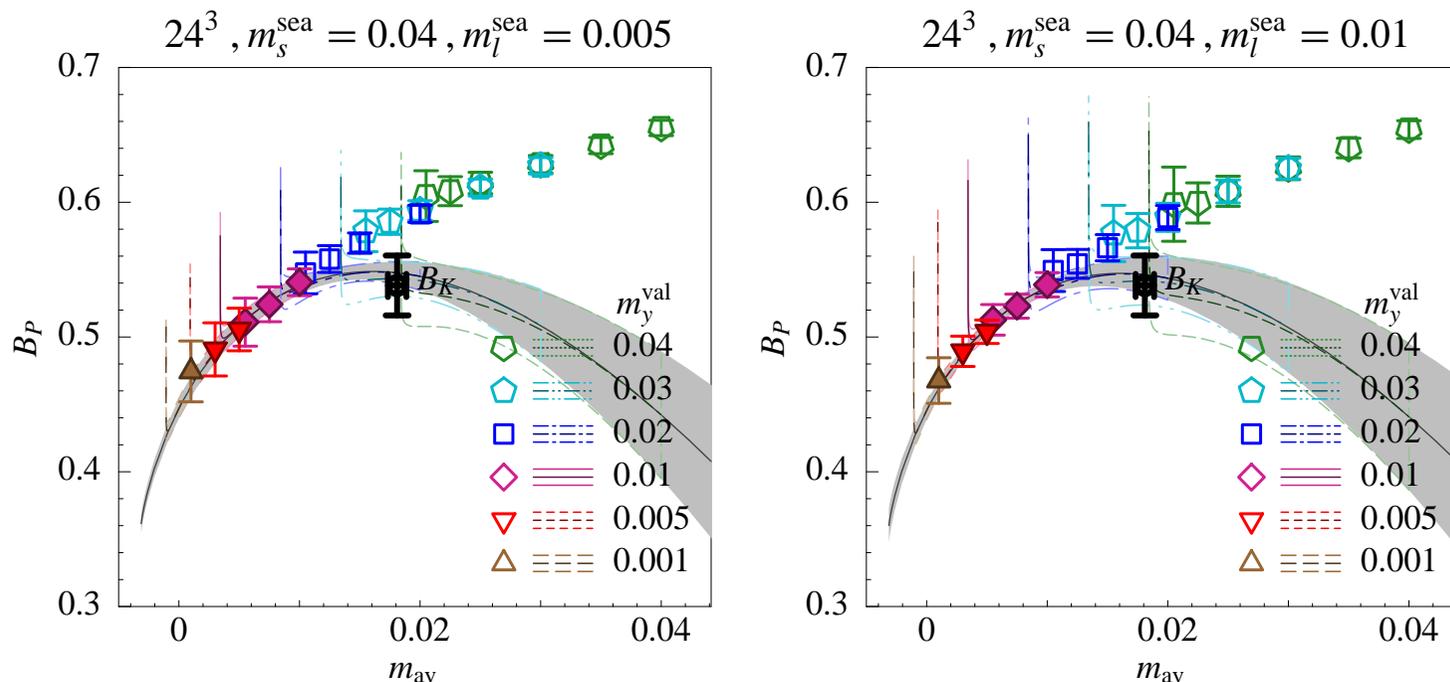


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$B_P$  2+1f Partially Quenched Chiral Fit



# Summary and Outlook

## Summary

Domain-wall fermions have made precision calculation of weak matrix elements such as the kaon bag parameter possible.

- Single scale (1.6 GeV), single volume (2 fm): [hep-ph/0702042](https://arxiv.org/abs/hep-ph/0702042)

$$B_K^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.557(12)(16)$$

## Outlook

- Second volume (3 fm) well underway:

$$B_K^{\text{lat}} = 0.578(10)$$

- Second scale (2.1 GeV) and continuum extrapolation soon to follow.

# More Information



## Further Reading

For further details on our methodology using older  $L_s = 8$  data, please see:

- First results from 2+1-flavor domain-wall QCD: mass spectrum, topology change and chiral symmetry with  $L_s = 8$ , [hep-lat/0612005](#)

For more details on the data presented here, please see the forthcoming:

- Light meson spectroscopy with 2+1 flavors of dynamical domain-wall fermions, [hep-lat/0701013](#)
- Neutral kaon mixing from 2+1 flavor domain-wall QCD, [hep-ph/0702042](#)

