Low-energy couplings from Lattice QCD in $\epsilon$-regime

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QCD and $\chi$-symmetry

The interactions of the light mesons at low momenta are determined to a great extent by the pattern of chiral symmetry breaking and therefore the QCD $\chi$-Lagrangian is a very useful tool in light hadron phenomenology.

Weinberg, Gasser and Leutwyler

$$\mathcal{L}^{QCD}_\chi = \mathcal{L}^{(2)}_\chi + \mathcal{L}^{(4)}_\chi + \ldots$$

$$\mathcal{L}^{(2)}_\chi = \frac{F^2}{4} \text{Tr} \left[ \partial_\mu U^\dagger \partial_\mu U \right] - \frac{\Sigma}{2} \text{Tr} \left[ e^{i\theta/N_f} M U + U^\dagger M^\dagger e^{-i\theta/N_f} \right]$$

$$\mathcal{L}^{(4)}_\chi = L_1 \text{Tr} \left[ \partial_\mu U^\dagger \partial_\mu U \right]^2 + L_2 \left( \text{Tr} \left[ \partial_\mu U^\dagger \partial_\nu U \right] \right)^2 + \ldots$$

The weak interactions responsible for weak decays as $\Delta S = 1, 2$ can also be included and parametrized in terms of more constants.

The lattice is the best non-perturbative method to bring in the missing information.
We need to do a precise matching of lattice QCD and $\chi$ Lagrangian:

- because it is practical: parametrize non-perturbative dynamics in a minimum set of LECs
- because the (lattice) world is never perfect...

<table>
<thead>
<tr>
<th>Theoretically</th>
<th>Practically</th>
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</thead>
<tbody>
<tr>
<td>$Ma \ll 1$</td>
<td>$Ma \geq 0.1$</td>
</tr>
<tr>
<td>$M_\pi L \gg 1$</td>
<td>$M_\pi L &lt; 5$</td>
</tr>
<tr>
<td>$\Lambda_{QCD} L \gg 1$</td>
<td>$\Lambda_{QCD} L \sim 2 - 3$</td>
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A precise matching of Lattice QCD and $\chi$PT is a necessary milestone in Lattice QCD
The need for exact $\chi$-symmetry

Wilson-type regularizations: $\chi$-breaking $O(a^2)$ can change the vacuum structure in a non-universal way when $m \leq a\Lambda^2_{QCD}$ or $m \leq a^2\Lambda^2_{QCD}$:

$$\lim_{m \to 0, V \to \infty} \lim_{a \to 0}$$

Empirically in unquenched simulations with Wilson fermions $M_\pi L \gg 1$ to avoid arbitrary small eigenvalues ...

Del Debbio, et al

Thanks to Ginsparg-Wilson regularizations the matching can be done at finite $a$:

\[ \Sigma(a) = \Sigma + O(a^2) \quad F(a) = F + O(a^2) \quad L_i(a) = L_i + O(a^2) \ldots \]

Ginsparg-Wilson, Kaplan, Shamir, Hasenfratz et al, Neuberger, Lüscher
Small $\mathcal{O}(a^2)$ violations

Scaling studies for overlap fermions show that $\mathcal{O}(a^2)$ corrections are small for several quantities such as $F_K$, $\Sigma$.
What $\chi$-regime?

$$\lim_{m \to 0} \lim_{V \to \infty} \leftrightarrow M_\pi L \gg 1$$

Finite-size corrections are strongly suppressed and only $m$ dependences remain...

$\chi$PT does not only predict the $m$ corrections but also those of finite $L$!

$p$-regime:

$$\lim_{m \to 0} \lim_{V \to \infty} \left| M_\pi L \geq O(1) \right.$$  

$\epsilon$-regime:

$$\lim_{m \to 0} \lim_{V \to \infty} \left| m \Sigma V = O(1) \right.$$
Finite-size scaling in $\chi$PT

In a finite volume we can distinguish two regimes of $\chi$PT:

$p$-regime: $m\Sigma V \gg 1$

$\epsilon$-regime: $m\Sigma V \leq 1$

Standard $\chi$PT in finite $V$:  

$m \sim p^2 \quad L^{-1}, T^{-1} \sim p$

$m, L$ effects sizeable

Zero-modes of pions are not perturbative!

$m \sim p^4 \quad L^{-1}, T^{-1} \sim p$

Only $L$ effects sizeable
Finite-size scaling in $\chi PT$

$V = \infty$ \hspace{1cm} $M_{\pi}L \geq O(1)$ \hspace{1cm} $m\Sigma V \leq O(1)$

$C_\infty(m, \Sigma, F, L_i, ...)$ \hspace{1cm} $C_p(m, L, \Sigma, F, L_i, ...)$ \hspace{1cm} $C_\epsilon(m, L, \Sigma, F, L_i, ...)$
Implies a reordering of the $\chi$ expansion: at any order less relevant couplings appear as compared to the usual chiral expansion

<table>
<thead>
<tr>
<th>Gasser, Leutwyler; Hansen; Hansen, Leutwyler; Damgaard, et al; PH, Laine</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLO: $\chi \equiv MU$ $\mathcal{L}<em>\mu \equiv i\partial</em>\mu UU^\dagger$, $\mathcal{W}<em>{\mu\nu} = 2(\partial</em>\mu \mathcal{L}<em>\nu + \partial</em>\nu \mathcal{L}<em>\mu)$; $(\Delta</em>{ij})<em>{ab} = \delta</em>{ai}\delta_{bj}$</td>
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<tr>
<td>$p$-regime</td>
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<tr>
<td>$L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle$</td>
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<td>$L_5 \langle D_\mu U^\dagger D^\mu U \left( U^\dagger \chi + \chi^\dagger U \right) \rangle$</td>
</tr>
<tr>
<td>$L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2$</td>
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<tr>
<td>$L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle$</td>
</tr>
<tr>
<td>$\mathcal{H}^{SU(4)}_{weak}$</td>
</tr>
<tr>
<td>Kambor, Missimer, Wyler</td>
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<tr>
<td>$D^\pm t_{ij,kl} \langle \Delta_{ij}(\chi - \chi^\dagger) \rangle \langle \Delta_{kl}(\chi - \chi^\dagger) \rangle$</td>
</tr>
<tr>
<td>$D^\pm t_{ij,kl} \langle \Delta_{ij} \mathcal{L}<em>\mu \rangle \langle \Delta</em>{kl} { \mathcal{L}^\mu, (\chi + \chi^\dagger) } \rangle$</td>
</tr>
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<td>$D^\pm t_{ij,kl} \langle \Delta_{ij} \mathcal{L}<em>\mu \rangle \langle \Delta</em>{kl} \mathcal{L}_\mu \rangle \langle (\chi + \chi^\dagger) \rangle$</td>
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<td>$D^\pm t_{ij,kl} \langle \Delta_{ij} \mathcal{L}<em>\mu \rangle \langle \Delta</em>{kl} \partial_\nu \mathcal{W}_{\mu\nu} \rangle$</td>
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<tr>
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</tr>
</tbody>
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The role of topology in $\epsilon$-regime

Correlation functions depend on topology in the $\epsilon$-regime:

Leutwyler, Smilga (1992)

- Poles $1/m$ in quark propagators

$$D_{xy}^{-1} = \sum_{i, \text{zero modes}} \frac{v_i(x)v_i(y)\dagger}{mV} + \ldots$$

- Non-zero modes are repelled by the zero modes: For $\lambda \to 0$:

$$\rho_\nu(\lambda) \sim \lambda^{2(|\nu|+N_f)+1}$$

We can consider averages on fixed-topological sectors and the $|\nu|$ dependence becomes a third scaling variable: $\chi$PT should reproduce this dependence

**$\epsilon$-regime:** matching of $L, |\nu|$ dependences

**$p$-regime:** matching of $m, L$ dependence
The numerical Challenge

In the $\epsilon$-regime: $m \Sigma V \leq 1$, large fluctuations in the observables are observed:

$$\langle \lambda_i \rangle \nu = \mathcal{O}(1) \frac{\Sigma V}{\Sigma V}, \quad \Delta \lambda = \lambda_{i+1} - \lambda_i \sim \mathcal{O}(1) \frac{\Sigma V}{\Sigma V} \geq m$$

Low-lying spectrum of $D_m$ is discrete: $\Delta \lambda \geq \lambda_k + m$
Space-time fluctuations in the wave-functions of the low-lying spectrum → large fluctuations in point-to-all propagators!

Two strategies to tame these fluctuations:

- **Low-mode averaging**: treat low-modes separately

\[
S(x, y) = S_h(x, y) + S_l(x, y), \quad S_l(x, y) = \frac{1}{V} \sum_{k=1}^{N_{\text{low}}} \frac{v_k(x)v_k(y)}{\lambda_k + m}
\]

- **Physics from zero-mode wave-functions**: use topological zero modes as probes

Giusti, PH, Laine, Weisz, Wittig
\textbf{\(\epsilon\)-regime simulations}

A number of \(\epsilon\)-regime simulations have been performed in the quenched approximation up to volumes of \(L \sim 2\) fm

- Condensate: \(\Sigma\)

\[
\text{PH, Jansen, Lellouch; Degrang; Degrang, Schaefer; Giusti, Necco}
\]

\(\rightarrow\) Necco’s talk

- Two-point functions: strong LECs such as \(F, \alpha_5, \ldots\)

\[
\text{Degrang, Schaefer; Bietenholz, \textit{et al}; Giusti et al, Gattringer et al, BGR coll.; Fukaya et al}
\]

- Three-point functions: weak LECs

\[
\text{Giusti et al.}
\]
$L$ and $|\nu|$ scaling of current correlators

$p$-regime (NLO): \[ Z^2_J \sum_{x} \langle J^a_L(x) J^b_L(0) \rangle_{\nu} \simeq -\text{Tr}[T^a T^b] \frac{F^2_{\nu} M_{\nu}}{8} \frac{\cosh(M_{\nu}(T/2-x_0))}{\sinh(M_{\nu}T/2)} \]

\[
F_L = F \left(1 + \frac{\alpha_5}{2} \frac{M^2}{(4\pi F)^2} \right)
\]

\[
M^2_{\nu} = \frac{2m \Sigma}{F^2} \left[1 - \frac{m_0^2}{(4\pi F)^2} \left(\ln \frac{M_{\nu}^2}{\mu^2} + 1\right) \right.
\]
\[ + \frac{M^2_{\nu}}{(4\pi F)^2} \left(\alpha \left(2 \ln \frac{M_{\nu}^2}{\mu^2} + 1\right) + 2\alpha_8(\mu) - \alpha_5(\mu) \right) + f(M, L) \]

$\epsilon$-regime (NLO) \[ Z^2_J \sum_{x} \langle J^a_L(x) J^b_L(0) \rangle_{\nu} = \text{Tr}[T^a T^b] \left(\frac{F^2_{\nu}}{2T} + \frac{\mu \Sigma_{\nu}(\mu)}{L^3} h_1 \left(\frac{x_0}{T}\right) \right) \]

\[
\Sigma_{\nu}(\mu) \equiv \Sigma \left[\mu \left(I_{\nu}(\mu) K_{\nu}(\mu) + I_{\nu+1}(\mu) K_{\nu-1}(\mu)\right) + \frac{\nu}{\mu} \right] \quad \mu \equiv m \Sigma V
\]

\[
h_1(\tau) \equiv \frac{1}{2} \left[\left( |\tau| - \frac{1}{2} \right)^2 - \frac{1}{12} \right]
\]
Test in $V = 16^4/16^3 32, \beta = 6.0, 0 \leq |\nu| \leq 2$

With the use of low-mode averaging we could simulate the $\epsilon$-regime

Giusti, PH, Laine, Weisz, Wittig 2004
$L$ and $|\nu|$ scaling of current correlators

<table>
<thead>
<tr>
<th>lattice</th>
<th>$\beta$</th>
<th>$V/a^4$</th>
<th>$r_0/a$</th>
<th>$L[\text{fm}]$</th>
<th>Conf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A$_0$</td>
<td>5.8458</td>
<td>$12^4$</td>
<td>4.026</td>
<td>1.49</td>
<td>388</td>
</tr>
<tr>
<td>A$_1$</td>
<td>6.0</td>
<td>$16^4$</td>
<td>5.368</td>
<td>1.49</td>
<td>596</td>
</tr>
<tr>
<td>B$_0$</td>
<td>5.8458</td>
<td>$16^4$</td>
<td>4.026</td>
<td>1.96</td>
<td>380</td>
</tr>
<tr>
<td>C$_0$</td>
<td>5.8485</td>
<td>$16^332$</td>
<td>4.026</td>
<td>1.96</td>
<td>826</td>
</tr>
</tbody>
</table>

PH, S. Necco et al, in preparation
LECs from zero-mode wave functions


- For correlators computed in fixed topological sectors exact poles in $1/m^n$ may appear when some propagators are saturated by the zero modes

$$D_{xy}^{-1} = \sum_{i, \text{zero modes}} \frac{v_i(x)v_i(y)^\dagger}{mV} + \ldots$$

- The residuals of these poles are better conditioned in the IR than the correlators themselves

One can match the residuals instead of the correlator in the $\epsilon$-regime!

$$C_\nu(x - z, y - z) = \frac{Res_n}{(mV)^n} + \ldots \quad Res_n = \lim_{m \to 0} (mV)^n C_\nu(x - z, y - z)$$
F from zero modes wavefunctions

Consider the pseudoscalar density correlator in a topological sector of charge $\nu$:

$$C^I_J(x - y) = \left\langle P^I(x)P^J(y) \right\rangle_{\nu}, \quad P^I(x) \equiv \bar{\Psi}(x)T^I\gamma_5\Psi(x)$$

The spectral decomposition of the propagators shows a pole in $1/m^2$ exists:

$$\lim_{m \to 0} (mV)^2 C^I_J(x) = \text{tr}[T^IT^J] A_\nu(x) + \text{tr}[T^I] \text{tr}[T^J] \tilde{A}_\nu(x)$$

$$A_\nu(x - y) \equiv \left\langle \sum_{i,j \in K} v^\dagger_j(x)v_i(x)v^\dagger_i(y)v_j(y) \right\rangle_{\nu} \quad \tilde{A}_\nu(x - y) \equiv -\left\langle \sum_{i \in K} v^\dagger_i(x)v_i(x) \sum_{j \in K} v^\dagger_j(y)v_j(y) \right\rangle_{\nu}$$

$$Dv_i = 0, \quad \forall \ i \in K \quad \text{and} \quad \int d^4x \ v^\dagger_i(x)v_i(x) = V.$$
F from zero modes wavefunctions

Remarkably in the $\epsilon$-regime of $\chi$PT we find the same pole!

Matching at LO ($h_1(\tau) \equiv \frac{1}{2} \left[ (\tau - \frac{1}{2})^2 - \frac{1}{12} \right]$, $\tau \equiv t/T$):

$$A(t) = \int d^3x \left\langle \sum_{i,j \in K} v_j^\dagger(x)v_i(x)v_i^\dagger(0)v_j(0) \right\rangle_\nu \simeq |\nu|L^3 + \frac{2|\nu|}{N_fF^2}(1 + N_f|\nu|)Th_1(\tau)$$

$$\tilde{A}(t) = \int d^3x \left\langle \sum_{i \in K} v_i^\dagger(x)v_i(x)\sum_{j \in K} v_j^\dagger(0)v_j(0) \right\rangle_\nu \simeq -\nu^2L^3 - \frac{2|\nu|}{N_fF^2}(N_f + |\nu|)Th_1(\tau)$$

At NLO, still the temporal dependence $(A'(t), \tilde{A}'(t))$ only depends on $F$!

In qChPT instead the singlet couplings enter at NLO: $\alpha, m_0...$
Compared with the use of left-current correlators:

- No need to compute the low eigenvalues/functions, only $|\nu|$ zero modes
- No need to compute propagators
- All-to-all correlator automatic
- Deep $\epsilon$-regime accesible

A test in the quenched approximation:

| lattice | $\beta$   | $L/a$ | $r_0/a$ | $L[fm]$ | $N_{\text{meas}}(|\nu| = 1)$ | $N_{\text{meas}}(|\nu| = 2)$ |
|---------|-----------|------|---------|---------|-------------------------------|-------------------------------|
| $B_0$   | 5.8458    | 12   | 4.026   | 1.49    | 880                           | 696                           |
| $B_1$   | 6.0       | 16   | 5.368   | 1.49    | 307                           | 226                           |
| $B_2$   | 6.1366    | 20   | 6.710   | 1.49    | 326                           | 213                           |
| $C_0$   | 5.8784    | 16   | 4.294   | 1.86    | 229                           | 186                           |
| $C_1$   | 6.0       | 20   | 5.368   | 1.86    | 83                            | 78                            |
We found the quenched value $F = 115(7)\text{MeV}(r_0)$
$L$ and $|\nu|$ scaling

\[
D_\nu = L^2 A''(T/2), \quad \tilde{D}_\nu = L^2 \tilde{A}''(T/2)
\]

Solid: $N_f = 2$, Dashed: $N_f = 3$, Dotted: $N_f = 0$
A different look at the $\Delta I = 1/2$ rule


Lattice QCD can investigate in a well defined way the role of the different scales that enter in the problem, in particular the role of $m_c$: if the large enhancement is due to the large separation between $m_c \gg \Lambda_{QCD}$ or $m_c \gg m_u$ there should be no effect in the theory with a light charm quark!

\[
\begin{align*}
  m_c & \quad \text{QCD} \\
  \Lambda_{\chi PT} & \quad \text{ChPT} \\
  m_u = m_d = m_s & \quad \text{SU(4)} \\
  m_c^{\text{phys}} & \quad \text{SU(3)}
\end{align*}
\]
Weak effective couplings in $SU(4)$ limit

Two four quark operator $O^\pm$ in the $(84,1)$ and $(20,1)$

\[
H_{w}^{\text{ChPT}} = \frac{g_{w}^2}{4M_{W}^{2}}(V_{us})^{*}V_{ud} \sum_{\sigma=\pm} g^{\sigma}[O^{\sigma}]
\]

\[
O^\pm = \frac{F^4}{4} \left[ (U \partial_\mu U^{\dagger})_{us} (U \partial_\mu U^{\dagger})_{du} \pm (U \partial_\mu U^{\dagger})_{uu} (U \partial_\mu U^{\dagger})_{ds} - (u \rightarrow c) \right]
\]

In contrast with $SU(3)$, only two operators appear in $SU(4)$-ChPT at LO:

\[
\frac{A_0}{A_2} = \frac{1}{\sqrt{2}} \left( \frac{1}{2} + \frac{3}{2}g^- \right) \quad [g^+]_{N_c} = [g^-]_{N_c} = 1
\]
The Matching

We perform the matching by equating correlation functions of the weak operators and two left currents in lattice QCD and in the chiral theory, more concretely the ratios:

\[ R^\sigma(x_0, y_0) \equiv \frac{\sum_{x,y} \langle [J_{L0}(x)]_{\alpha\beta} O^\pm(0) [J_{L0}(y)]_{\gamma\delta} \rangle}{\sum_x \langle [J_{L0}(x)]_{\alpha\beta} [J_{L0}(0)]_{\beta\alpha} \rangle \sum_y \langle [J_{L0}(y)]_{\alpha\beta} [J_{L0}(0)]_{\beta\alpha} \rangle} \]

\[ g^\sigma [\mathcal{R}^\sigma(m, V, LECS)] = k^\sigma \left( \frac{M_W}{\Lambda} \right) \frac{Z^\sigma(g_0)}{Z^2_A} R^\sigma \]

\[ \downarrow \quad \chi PT \quad \downarrow \quad P.T. - 2 \ loop \quad \downarrow \quad N.P. \quad \downarrow \quad \text{Lattice} \]

In the \( \epsilon \)-regime at NLO: \( \mathcal{R}^\sigma(x_0, y_0) \) independent of \( x_0, y_0, |\nu| \) and any other LEC different from \( g^\pm \)
Tested in the quenched approximation:

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$L/a$</th>
<th>$T/a$</th>
<th>$n_{\text{low}}$</th>
<th>$L[\text{fm}]$</th>
<th>$m$</th>
<th>$# \text{cfgs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$-regime</td>
<td>5.8485</td>
<td>16</td>
<td>32</td>
<td>20</td>
<td>2</td>
<td>$m_s/40, m_s/60$</td>
<td>$O(800)$</td>
</tr>
<tr>
<td>$p$-regime</td>
<td>5.8485</td>
<td>16</td>
<td>32</td>
<td>20</td>
<td>2</td>
<td>$m_2/2 - m_s/6$</td>
<td>$O(200)$</td>
</tr>
</tbody>
</table>

The expected features of the $R^\sigma(x_0, y_0)$ in the $\epsilon$-regime: independence on $x_0, y_0, m$ and $\nu$ are well reproduced by the data.
$g^\pm$ in $SU(4)$-limit

\begin{tabular}{c|c|c}
 & $g^+$ & $g^-$ \\
This work & 0.51(3)(5)(6) & 2.6(1)(3)(3) \\
"Exp" & $\sim 0.5$ & $\sim 10.4$ \\
Large $N_c$ & 1 & 1 \\
\end{tabular}
We can define e.g.

\[ \tilde{R}_\sigma^\nu \equiv \frac{\lim_{m \to 0} (mV)^2 \sum_{x,y} \langle \partial_x P^a(x) O^\pm(0) \partial_y P^b(y) \rangle_\nu}{\lim_{m \to 0} (mV) \sum_x \langle \partial_x P^a(x) J_{L_0}^a(0) \rangle_\nu \lim_{m \to 0} (mV) \sum_y \langle \partial_y P^b(y) J_{L_0}^b(0) \rangle_\nu} \]

\[ \tilde{R}_\sigma^\nu = \frac{A_\nu(x_0, y_0) \pm \tilde{A}_\nu(x_0, y_0)}{B_\nu(x_0) B_\nu(y_0)} \]

\[ A_\nu(x_0, y_0) \equiv - \int d^3 x \int d^3 y \left\langle \sum_{i \in K} \eta_i^\dagger(x) S(x, 0) \gamma_\mu P - v_i(0) \sum_{j \in K} \eta_j^\dagger(y) S(y, 0) \gamma_\mu P - v_j(0) \right\rangle_\nu \]

\[ \tilde{A}_\nu(x_0, y_0) \equiv \int d^3 x \int d^3 y \left\langle \sum_{i,j \in K} \eta_j^\dagger(x) S(x, 0) \gamma_\mu P - v_i(0) \eta_i^\dagger(y) S(y, 0) \gamma_\mu P - v_j(0) \right\rangle_\nu \]

with \( \eta_i(x) \equiv \frac{v_i(x+a\hat{0})-v_i(x-a\hat{0})}{2} \)
Matching formula:

\[ g_\pm \mathcal{R}_\nu^\pm = \left[k^\pm(M_W)\right]_{RGI} \left[\frac{Z^\pm}{Z_\Lambda^2}\right]_{RGI} \mathcal{R}_\nu^\pm \]

- The renormalization factor is the same as before!
- At LO the \((q)\chi PT\) result is extremely simple:

\[ \mathcal{R}_\nu^\pm = (1 \mp |\nu|^{-1}) + ... \]
- NLO corrections are still quite large at 2 fm
Advantages with respect to left-current correlators:

- $2 \times |\nu| \text{ inversions versus } 12 + 2 \times N_{low}$ to get $\sum_{x,y,\bar{t},t_0} |x_0-t_0, y_0-t_0=\text{fixed}$

- Even if no low-modes are computed $\sum_{\bar{x},\bar{y},\bar{t}}$

- Completely different observable in $\epsilon$-regime: rather different chiral corrections

Exploratory Study:

$\beta = 5.8458, V/a^4 = 16^4, 1 \leq |\nu| \leq 5, m\Sigma V \sim O(1), N_{conf} = 282$
Exploratory Study: 2PT

At NLO in $\chi PT$: \[ B_\nu(x_0) = \alpha_\nu + \beta_\nu \left(\frac{x_0}{T} - \frac{1}{2}\right)^2 \]
scaling of 2PT

$$\alpha_\nu = |\nu| - \frac{1}{12} \frac{T}{L (F L)^2} |\nu|^2 \quad \beta_\nu = \frac{T}{L (F L)^2} |\nu|^2$$

The value of $F$ from NLO terms is in reasonable agreement with other determinations.
Chiral Ward Identity from zero modes

The Ward identity relates the JP and PP correlators:

$$Z_A B_\nu(x_0 - y_0) = \lim_{m \to 0} m^2 V \int d^3x \langle \mathcal{P}^a(x) \mathcal{P}^a(y) \rangle_\nu$$

Using the standard $Z_A$ determination
There is a nice signal for $\overline{R}_\nu^-$, less clear for $\overline{R}_\nu^+$

Things seem to work as expected from the numerical point of view...but need longer time extent: work in progress!
Summary

• Fermion regularizations with exact chiral symmetry should allow us to do a precise matching to $\chi$PT

• $\chi$PT can predict not just the scaling with the quark mass, but also with $L$ and $|\nu|$ scaling in the $\epsilon$-regime is less affected by unknown higher order LECs than the scaling with $m$

• Many simulations have been performed in the $\epsilon$-regime in the quenched approximation: condensate, two and three-point functions and things seem to work as expected...

• $\epsilon$-regime simulations in the unquenched theory are becoming feasible...