Low-energy couplings from Lattice QCD in ϵ-regime

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QCD and χ-symmetry

The interactions of the light mesons at low momenta are determined to a great extent by the pattern of chiral symmetry breaking and therefore the QCD χ-Lagrangian is a very usefull tool in light hadron phenomenology

Weinberg, Gasser and Leutwyler

$$\mathcal{L}_\chi^{QCD} = \mathcal{L}_\chi^{(2)} + \mathcal{L}_\chi^{(4)} + \ldots$$

$$\begin{aligned}
\mathcal{L}_\chi^{(2)} &= \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[e^{i\theta/N_f} M U + U^\dagger M^\dagger e^{-i\theta/N_f}\right] \\
\mathcal{L}_\chi^{(4)} &= L_1 \mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right]^2 + L_2 \left(\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\nu U\right]\right)^2 + \ldots
\end{aligned}$$

The weak interactions responsible for weak decays as $\Delta S = 1, 2$ can also be included and parametrized in terms of more constants

The lattice is the best non-pertubative method to bring in the missing information

We need to do a precise macthing of lattice QCD and χ Lagrangian:

- because it is practical: parametrize non-perturbative dynamics in a minimum set of LECs

- because the (lattice) world is never perfect...

Theoretically	Practically
$Ma \ll 1$	$Ma \geq 0.1$
$M_\pi L \gg 1$	$M_\pi L < 5$
$\Lambda_{QCD} L \gg 1$	$\Lambda_{QCD} L \sim 2 - 3$

A precise matching of Lattice QCD and χPT is a necessary milestone in Lattice QCD

The need for exact χ-symmetry

Wilson-type regularizations: χ-breaking $\mathcal{O}(a^2)$ can change the vaccuum structure in a non-universal way when $m \leq a\Lambda_{QCD}^2$ or $m \leq a^2\Lambda_{QCD}^2$:

$$\lim_{m\to 0, V\to\infty} \lim_{a\to 0}$$

Empirically in unquenched simulations with Wilson fermions$M_\pi L \gg 1$ to avoid arbitrary small eigenvalues ...

Del Debbio, et al

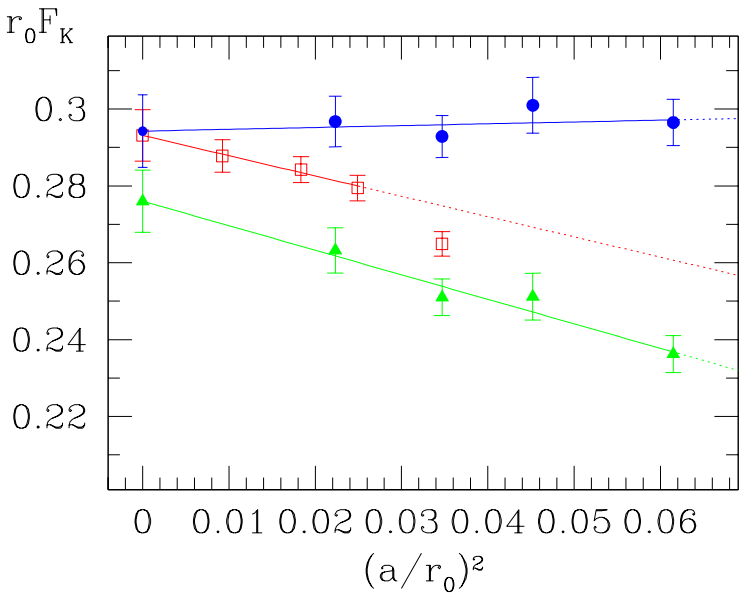
Thanks to Ginsparg-Wilson regularizations the matching can be done at finite a:

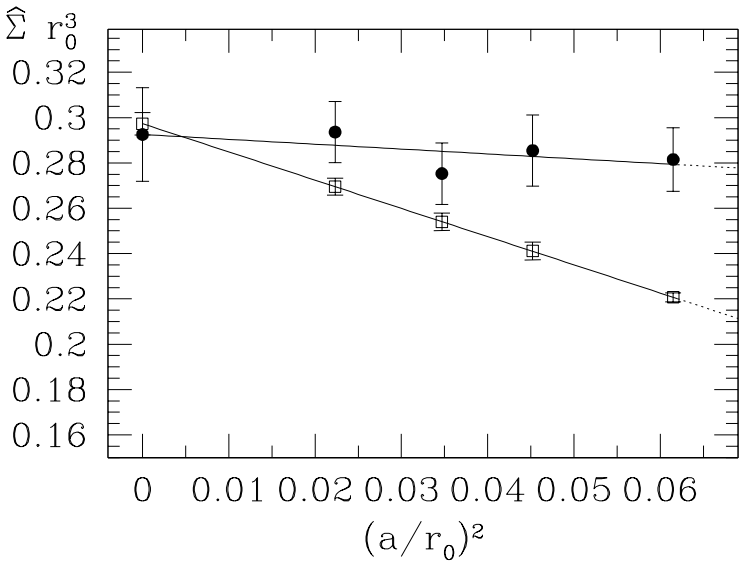
Ginsparg-Wilson, Kaplan, Shamir, Hasenfratz *et al*, Neuberger, Lüscher

$$\Sigma(a) = \Sigma + \mathcal{O}(a^2) \quad F(a) = F + \mathcal{O}(a^2) \quad L_i(a) = L_i + \mathcal{O}(a^2)...$$

Small $\mathcal{O}(a^2)$ violations

Scaling studies for overlap fermions show that $\mathcal{O}(a^2)$ corrections are small for several quantities such as F_K, Σ





Wennekers, Wittig

What χ-regime ?

$$\lim_{m\to 0} \lim_{V\to\infty} \leftrightarrow M_\pi L \gg 1$$

Finite-size corrections are strongly suppressed and only m dependences remain...

χPT does not only predict the m corrections but also those of finite L !

p-regime:

$$\lim_{\substack{m\to 0\\ V\to\infty}} \Big|_{M_\pi L \geq \mathcal{O}(1)}$$

ϵ-regime:

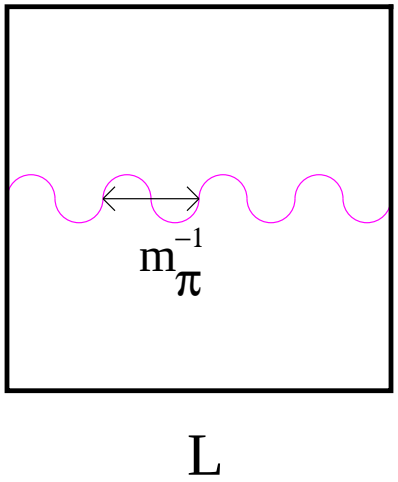
$$\lim_{\substack{m\to 0\\ V\to\infty}} \Big|_{m\Sigma V = \mathcal{O}(1)}$$

Finite-size scaling in χPT

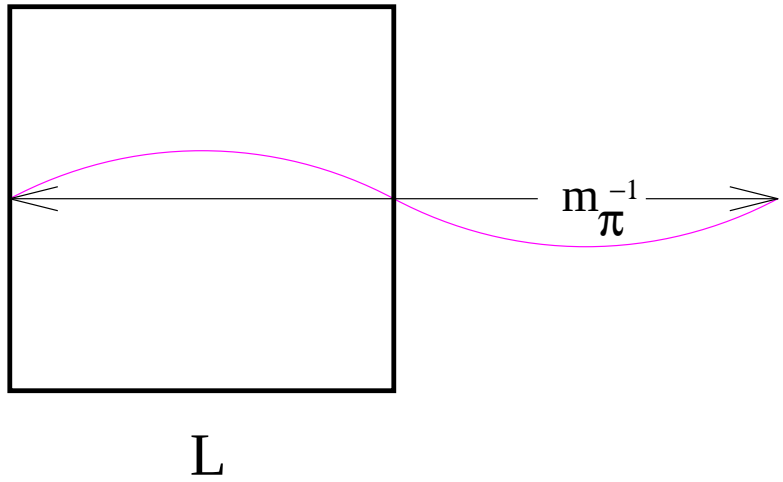
Gasser, Leutwyler

In a finite volume we can distinguish two regimes of χPT:

p-regime: $m\Sigma V \gg 1$



ϵ-regime: $m\Sigma V \leq 1$



Standard χPT in finite V:

$m \sim p^2 \qquad L^{-1}, T^{-1} \sim p$

m, L effects sizeable

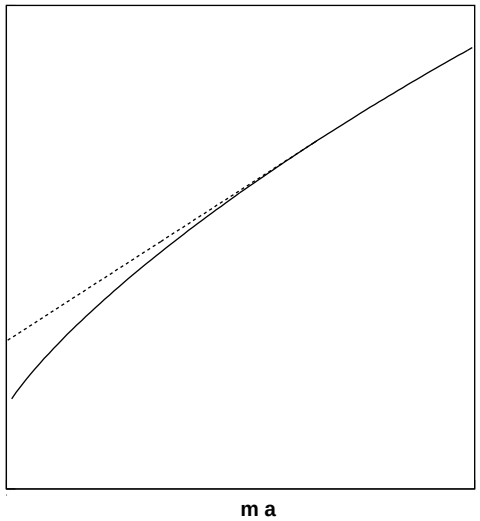
Zero-modes of pions are not perturbative!

$m \sim p^4 \qquad L^{-1}, T^{-1} \sim p$

Only L effects sizeable

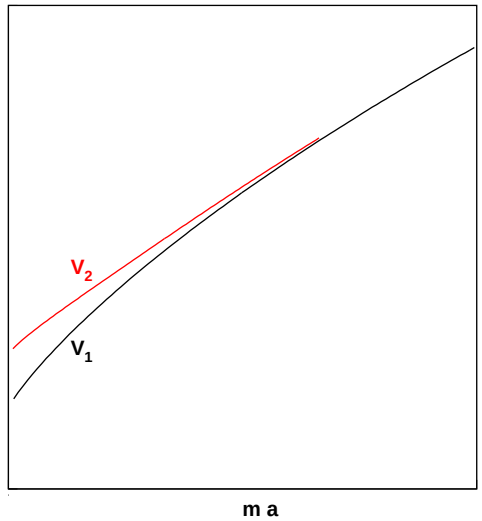
Finite-size scaling in χPT

$V = \infty$



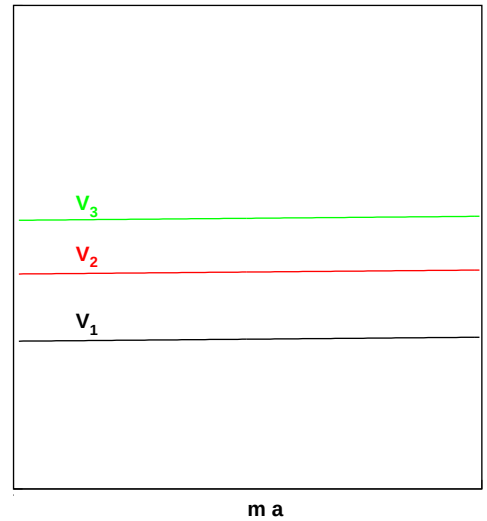
$C_\infty(m, \Sigma, F, L_i, ...)$

$M_\pi L \geq \mathcal{O}(1)$



$C_p(m, L, \Sigma, F, L_i, ...)$

$m\Sigma V \leq \mathcal{O}(1)$



$C_\epsilon(m, L, \Sigma, F, L_i, ...)$

Implies a reordering of the χ expansion: at any order less relevant couplings appear as compared to the usual chiral expansion

Gasser, Leutwyler; Hansen; Hansen, Leutwyler; Damgaard, *et al*; PH, Laine

NLO: $\chi \equiv MU \quad \mathcal{L}_\mu \equiv i\partial_\mu U U^\dagger, \quad \mathcal{W}_{\mu\nu} = 2(\partial_\mu \mathcal{L}_\nu + \partial_\nu \mathcal{L}_\mu); \quad (\Delta_{ij})_{ab} = \delta_{ai}\delta_{bj}$

		p-regime	ϵ-regime
Gasser, Leytwyler	$\mathcal{H}_{QCD}$	$L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi^\dagger U\rangle$	$\times$
		$L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle$	$\times$
		$L_6 \langle U^\dagger \chi + \chi^\dagger U\rangle^2$	$\times$
		$L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle$	$\times$
Kambor,Missimer,Wyler	$\mathcal{H}^{SU(4)}_{weak}$	$D^\pm_2 t^\pm_{ij,kl} \langle \Delta_{ij}(\chi - \chi^\dagger)\rangle\langle \Delta_{kl}(\chi - \chi^\dagger)\rangle$	$\times$
		$D^\pm_4 t^\pm_{ij,kl} \langle \Delta_{ij}\mathcal{L}_\mu\rangle\langle \Delta_{kl}\left\{\mathcal{L}^\mu, (\chi + \chi^\dagger)\right\}\rangle$	$\times$
		$D^\pm_7 t^\pm_{ij,kl} \langle \Delta_{ij}\mathcal{L}_\mu\rangle\langle \Delta_{kl}\mathcal{L}_\mu\rangle\langle (\chi + \chi^\dagger)\rangle$	$\times$
		$D^\pm_{20} t^\pm_{ij,kl} \langle \Delta_{ij}\mathcal{L}_\mu\rangle\langle \Delta_{kl}\partial_\nu \mathcal{W}_{\mu\nu}\rangle$	$\times$
		$D^\pm_{24} t^\pm_{ij,kl} \langle \Delta_{ij}\mathcal{W}_{\mu\nu}\rangle\langle \Delta_{kl}\mathcal{W}_{\mu\nu}\rangle$	$\times$

The role of topology in ϵ-regime

Correlation functions depend on topology in the ϵ-regime:

Leutwyler, Smilga (1992)

- Poles $1/m$ in quark propagators $D^{-1}_{xy} = \sum_{i,zero\ modes} \frac{v_i(x)v_i(y)^\dagger}{mV} + ...$
- Non-zero modes are repelled by the zero modes: For $\lambda \to 0$:

$$\rho_\nu(\lambda) \sim \lambda^{2(|\nu|+N_f)+1}$$

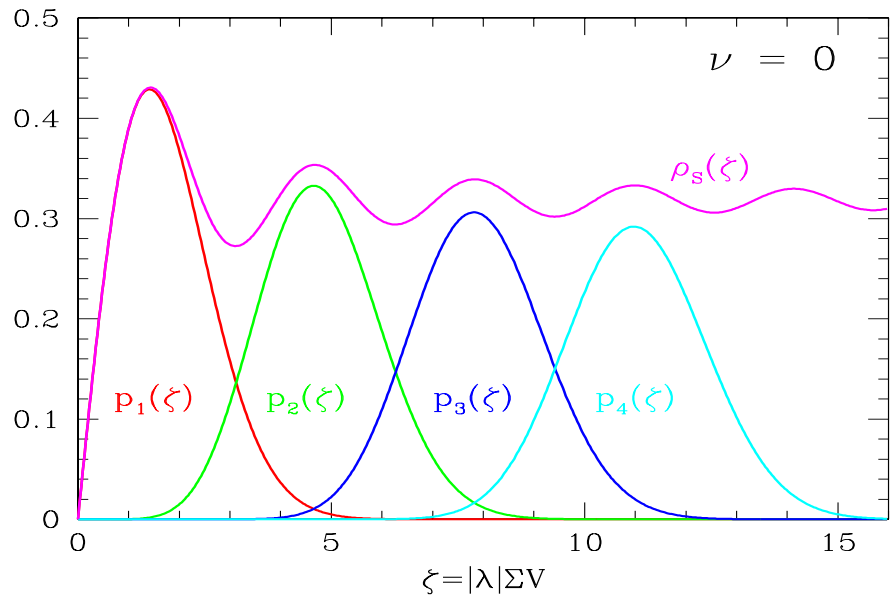
We can consider averages on fixed-topological sectors and the $|\nu|$ dependence becomes a third scaling variable: χPT should reproduce this dependence

ϵ-regime:matching of $L, |\nu|$ dependences

p-regime:matching of m, L dependence

The numerical Challenge

In the ϵ-regime: $m\Sigma V \leq 1$, large fluctuations in the observables are observed:



$$\langle \lambda_i \rangle_\nu = \frac{\mathcal{O}(1)}{\Sigma V}, \quad \Delta\lambda = \lambda_{i+1} - \lambda_i \sim \frac{\mathcal{O}(1)}{\Sigma V} \geq m$$

Low-lying spectrum of D_m is discrete: $\Delta\lambda \geq \lambda_k + m$

Space-time fluctuations in the wave-functions of the low-lying spectrum $\rightarrow$ large fluctuations in point-to-all propagators!

Two strategies to tame these fluctuations:

- Low-mode averaging : treat low-modes separately

$$S(x,y) = S_h(x,y) + S_l(x,y), \;\; S_l(x,y) = \frac{1}{V}\sum_{k=1}^{N_{low}} \frac{v_k(x)v_k(y)^\dagger}{\lambda_k + m}$$

Degrand, Schaefer; Giusti, PH, Laine, Weisz, Wittig

- Physics from zero-mode wave-functions : use topological zero modes as probes

Giusti, PH, Laine, Weisz, Wittig

ϵ-regime simulations

A number of ϵ-regime simulations have been performed in the quenched approximation up to volumes of $L \sim 2$ fm

- Condensate: Σ

PH,Jansen,Lellouch; Degrand; Degrand, Schaefer; Giusti,Necco

$\rightarrow$ Necco's talk

- Two-point functions: strong LECs such as $F, \alpha_5, ...$

Degrand, Schaefer; Bietenholz, *et al*; Giusti *et al*; Gattringer *et al*, BGR coll.; Fukaya *et al*

- Three-point functions: weak LECs

Giusti et al.

L and $|\nu|$ scaling of current correlators

p-regime (NLO): $$Z_J^2 \sum_{\vec{x}} \langle J_L^a(x) J_L^b(0) \rangle_\nu \simeq -\mathrm{Tr}[T^a T^b] \frac{F_L^2 M_L}{8} \frac{\cosh(M_L(T/2 - x_0))}{\sinh(M_L T/2)}$$

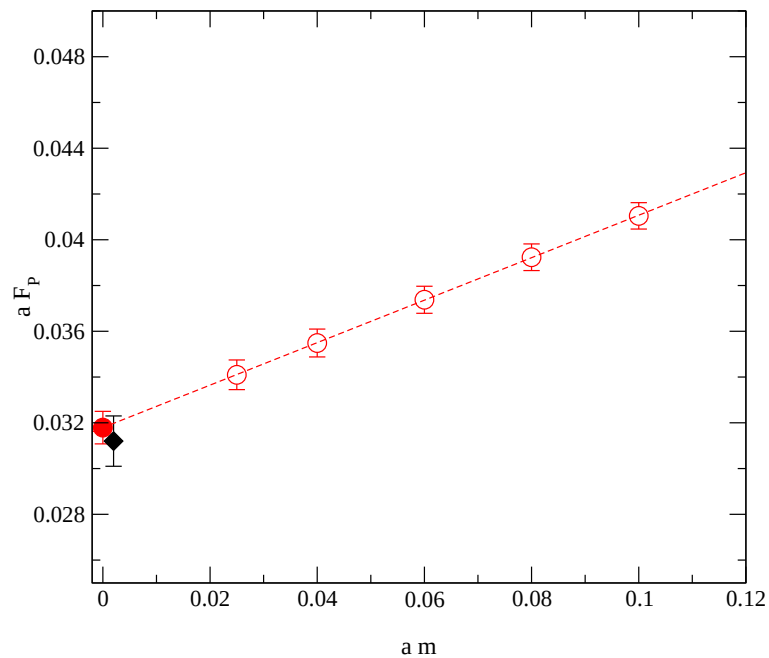
$$\begin{aligned} F_L &= F \left(1 + \frac{\alpha_5}{2} \frac{M^2}{(4\pi F)^2} \right) \\ M_L^2 &= \frac{2m\Sigma}{F^2} \left[1 - \frac{m_0^2}{(4\pi F)^2} \left(\ln \frac{M_P^2}{\mu^2} + 1 \right) \right. \\ &+ \left. \frac{M_P^2}{(4\pi F)^2} \left(\alpha \left(2 \ln \frac{M_P^2}{\mu^2} + 1 \right) + 2\alpha_8(\mu) - \alpha_5(\mu) \right) + f(M, L) \right] \end{aligned}$$

ϵ-regime (NLO) $$Z_J^2 \sum_{\vec{x}} \langle J_L^a(x) J_L^b(0) \rangle_\nu = \mathrm{Tr}[T^a T^b] \left(\frac{F^2}{2T} + \frac{\mu \Sigma_\nu(\mu)}{L^3} h_1 \left(\frac{x_0}{T} \right) \right)$$

$$\begin{aligned} \Sigma_\nu(\mu) &\equiv \Sigma \left[\mu \left(I_\nu(\mu) K_\nu(\mu) + I_{\nu+1}(\mu) K_{\nu-1}(\mu) \right) + \frac{\nu}{\mu} \right] \qquad \mu \equiv m\Sigma V \\ h_1(\tau) &\equiv \frac{1}{2} \left[\left(|\tau| - \frac{1}{2} \right)^2 - \frac{1}{12} \right] \end{aligned}$$

Test in $V = 16^4/16^3 32, \beta = 6.0, 0 \leq |\nu| \leq 2$

With the use of low-mode averaging we could simulation the ϵ-regime

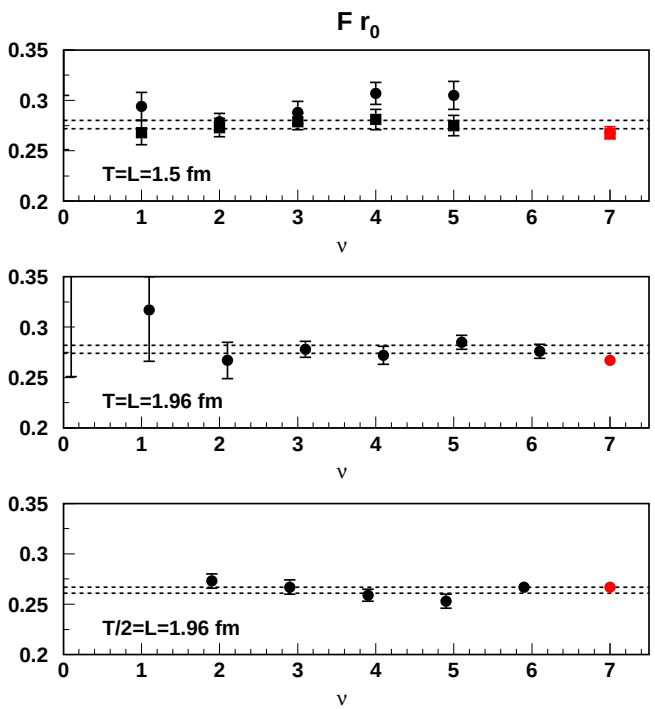


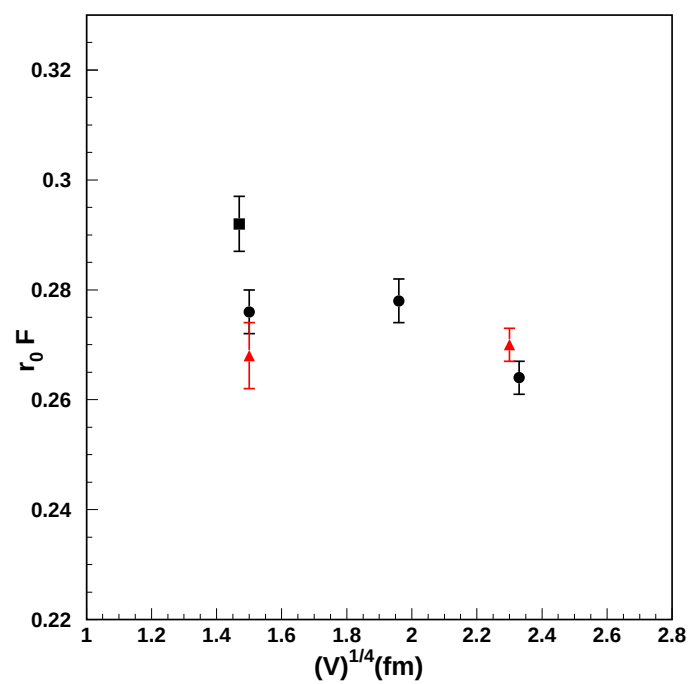
Giusti, PH, Laine, Weisz, Wittig 2004

L and $|\nu|$ scaling of current correlators

PH, S. Necco et al, in preparation

lattice	β	V/a^4	r_0/a	L[fm]	Confs
A_0	5.8458	12^4	4.026	1.49	388
A_1	6.0	16^4	5.368	1.49	596
B_0	5.8458	16^4	4.026	1.96	380
C_0	5.8485	$16^3 32$	4.026	1.96	826





LECs from zero-mode wave functions

Giusti, PH, Laine, Weisz, Wittig (2004)

- For correlators computed in fixed topological sectors exact poles in $1/m^n$ may appear when some propagators are saturated by the zero modes

$$D^{-1}_{xy} = \sum_{i,zero\ modes} \frac{v_i(x)v_i(y)^\dagger}{mV} + \ldots$$

- The residuals of these poles are better conditioned in the IR than the correlators themselves

One can match the residuals instead of the correlator in the ϵ-regime!

$$C_\nu(x-z, y-z) = \frac{Res_n}{(mV)^n} + \ldots \quad Res_n = \lim_{m\to 0} (mV)^n C_\nu(x-z, y-z)$$

F from zero modes wavefunctions

Consider the pseudoscalar density correlator in a topological sector of charge ν:

$$\mathcal{C}_\nu^{IJ}(x-y) = \left\langle P^I(x) P^J(y) \right\rangle_\nu, \; P^I(x) \equiv \bar{\Psi}(x) T^I \gamma_5 \Psi(x)$$

The spectral decomposition of the propagators shows a pole in $1/m^2$ exists:

$$\lim_{m\to 0} (mV)^2 \, \mathcal{C}_\nu^{IJ}(x) = \mathrm{tr}[T^I T^J] \, \mathcal{A}_\nu(x) + \mathrm{tr}[T^I] \, \mathrm{tr}[T^J] \, \tilde{\mathcal{A}}_\nu(x)$$

$$\mathcal{A}_\nu(x-y) \equiv \Big\langle \sum_{i,j\in\mathcal{K}} v_j^\dagger(x) v_i(x) v_i^\dagger(y) v_j(y) \Big\rangle_\nu \quad \tilde{\mathcal{A}}_\nu(x-y) \equiv - \Big\langle \sum_{i\in\mathcal{K}} v_i^\dagger(x) v_i(x) \sum_{j\in\mathcal{K}} v_j^\dagger(y) v_j(y) \Big\rangle_\nu$$

$$D v_i = 0, \; \forall \, i \in \mathcal{K} \text{ and } \int \mathrm{d}^4 x \, v_i^\dagger(x) v_i(x) = V.$$

F from zero modes wavefunctions

Remarkably in the ϵ-regime of χPT we find the same pole !

Matching at LO ($h_1(\tau) \equiv \frac{1}{2}\left[\left(\tau - \frac{1}{2}\right)^2 - \frac{1}{12}\right], \quad \tau \equiv t/T$):

$$A(t) = \int \mathrm{d}^3x \Big\langle \sum_{i,j\in\mathcal{K}} v_j^\dagger(x) v_i(x) v_i^\dagger(0) v_j(0) \Big\rangle_\nu \quad \simeq \quad |\nu| L^3 + \frac{2|\nu|}{N_\mathrm{f} F^2}(1 + N_\mathrm{f}|\nu|) T h_1(\tau)$$

$$\tilde{A}(t) = \int \mathrm{d}^3x \Big\langle \sum_{i\in\mathcal{K}} v_i^\dagger(x) v_i(x) \sum_{j\in\mathcal{K}} v_j^\dagger(0) v_j(0) \Big\rangle_\nu \quad \simeq \quad -\nu^2 L^3 - \frac{2|\nu|}{N_\mathrm{f} F^2}(N_\mathrm{f} + |\nu|) T h_1(\tau)$$

At NLO, still the temporal dependence $(A'(t), \tilde{A}'(t))$ only depends on F !

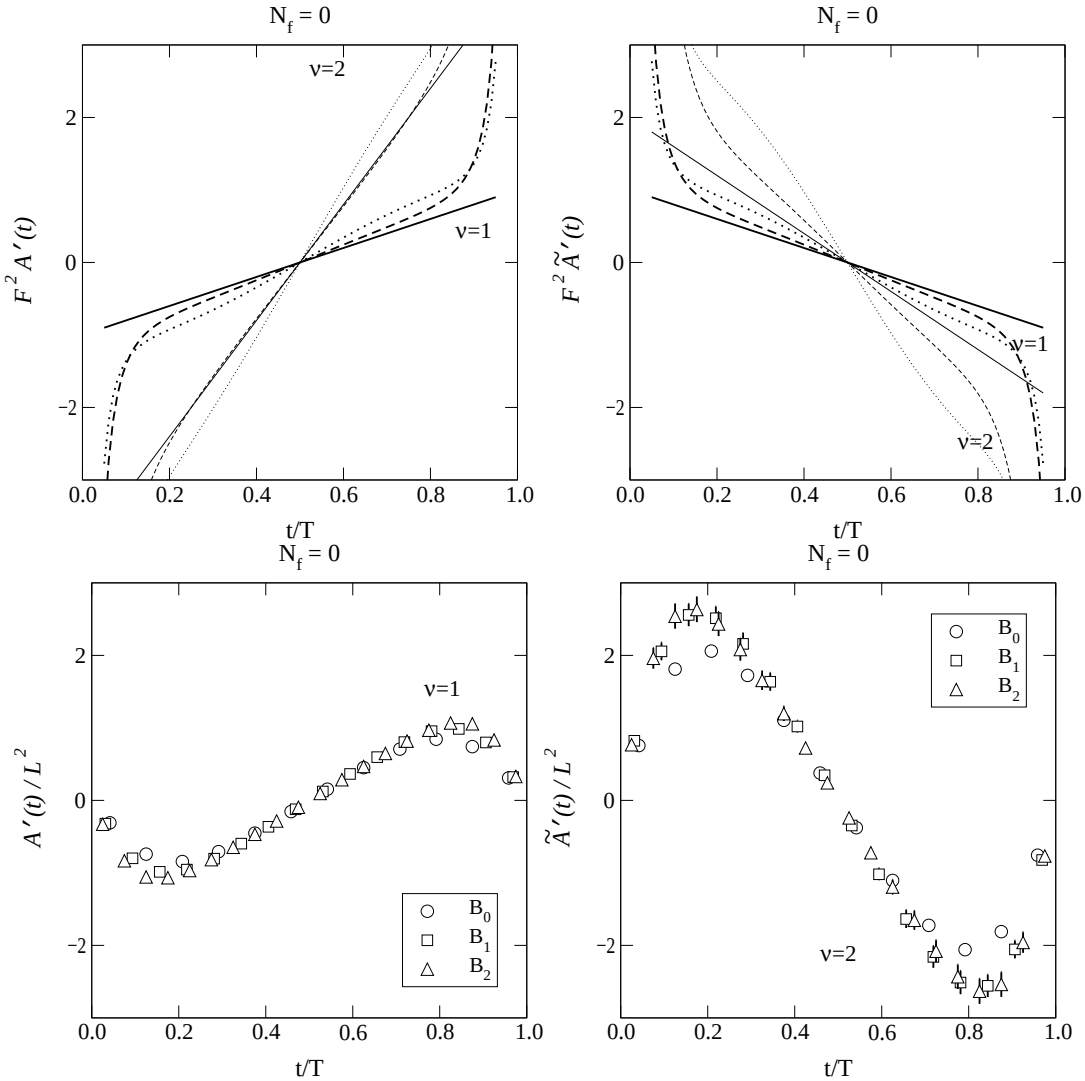
In qChPT instead the singlet couplings enter at NLO: α, m_0...

Compared with the use of left-current correlators:

- No need to compute the low eigenvalues/functions, only $|\nu|$ zero modes
- No need to compute propagators
- All-to-all correlator automatic
- Deep ϵ-regime accesible

A test in the quenched approximation:

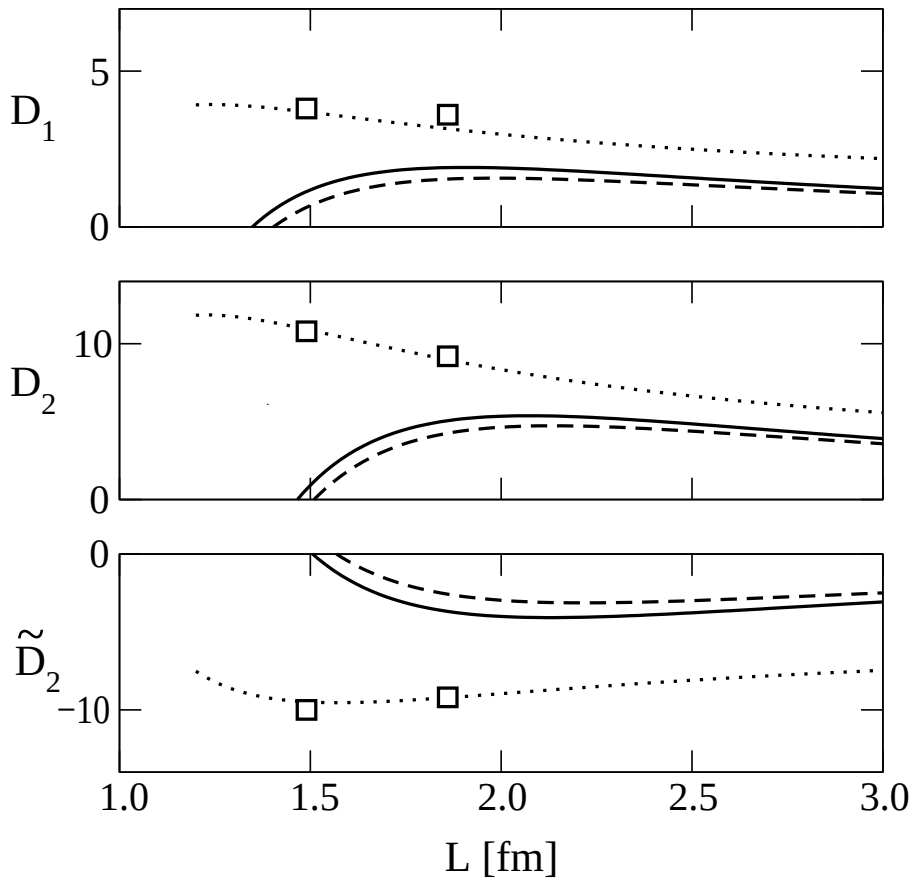
lattice	β	L/a	r_0/a	L[fm]	$N_{\rm meas}(\|\nu\|=1)$	$N_{\rm meas}(\|\nu\|=2)$
B_0	5.8458	12	4.026	1.49	880	696
B_1	6.0	16	5.368	1.49	307	226
B_2	6.1366	20	6.710	1.49	326	213
C_0	5.8784	16	4.294	1.86	229	186
C_1	6.0	20	5.368	1.86	83	78



We found the quenched value $F = 115(7)\mathrm{MeV}(r_0)$

L and $|\nu|$ scaling

$$D_\nu = L^2 A''(T/2), \tilde{D}_\nu = L^2 \tilde{A}''(T/2)$$

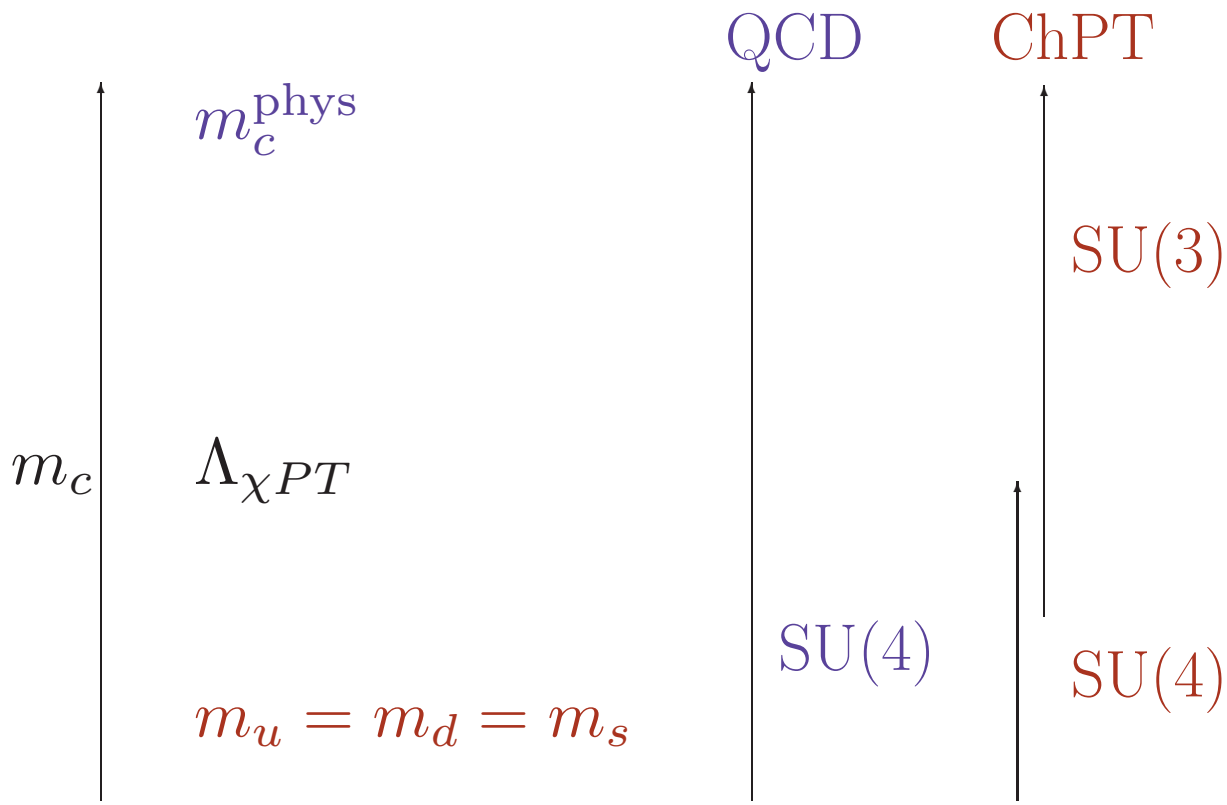


Solid: $N_f = 2$, Dashed: $N_f = 3$, Dotted: $N_f = 0$

A different look at the $\Delta I = 1/2$ rule

Lüscher and Giusti, PH, Laine, Weisz, Wittig (2004)

Lattice QCD can investigate in a well defined way the role of the different scales that enter in the problem, in particular the role of m_c: if the large enhancement is due to the large separation between $m_c \gg \Lambda_{QCD}$ or $m_c \gg m_u$ there should be no effect in the theory with a light charm quark!



Weak effective couplings in $SU(4)$ limit

Two four quark operator $O^{\pm}$ in the $(84,1)$ and $(20,1)$

$$\mathcal{H}_{\mathrm{w}}^{\mathrm{ChPT}} = \frac{g_{\mathrm{w}}^2}{4M_W^2}(V_{us})^* V_{ud} \sum_{\sigma=\pm} g^{\sigma}[\mathcal{O}^{\sigma}]$$

$$\mathcal{O}^{\pm} = \frac{F^4}{4}\left[\left(U\partial_\mu U^\dagger\right)_{us}\left(U\partial_\mu U^\dagger\right)_{du} \pm \left(U\partial_\mu U^\dagger\right)_{uu}\left(U\partial_\mu U^\dagger\right)_{ds} - (u \to c)\right]$$

In contrast with $SU(3)$, only two operators appear in $SU(4)$-ChPT at LO:

$$\frac{A_0}{A_2} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{3}{2}\frac{g^-}{g^+}\right) \qquad [g^+]_{N_c} = [g^-]_{N_c} = 1$$

The Matching

We perform the matching by equating correlation functions of the weak operators and two left currents in lattice QCD and in the chiral theory, more concretely the ratios:

$$R^\sigma(x_0, y_0) \equiv \frac{\sum_{\mathbf{x},\mathbf{y}} \langle [J_{L0}(x)]_{\alpha\beta}\, O^\pm(0)\, [J_{L0}(y)]_{\gamma\delta} \rangle}{\sum_{\mathbf{x}} \langle [J_{L0}(x)]_{\alpha\beta} [J_{L0}(0)]_{\beta\alpha} \rangle \sum_{\mathbf{y}} \langle [J_{L0}(y)]_{\alpha\beta} [J_{L0}(0)]_{\beta\alpha} \rangle}$$

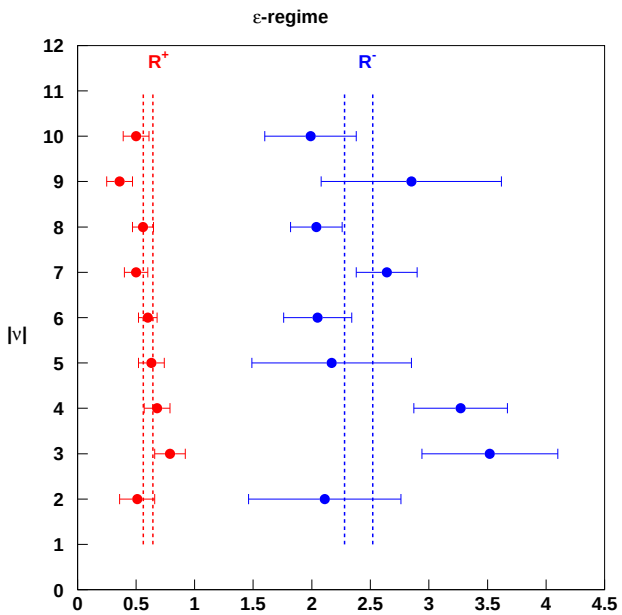
$$\begin{array}{ccccc} g^\sigma & [\mathcal{R}^\sigma(m, V, LECS)] = & k^\sigma(M_W/\Lambda) & \frac{Z^\sigma(g_0)}{Z_A^2} & R^\sigma \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & \chi PT & \mathrm{P}.T. - 2\ loop & \mathrm{N}.P. & \mathrm{L}attice \end{array}$$

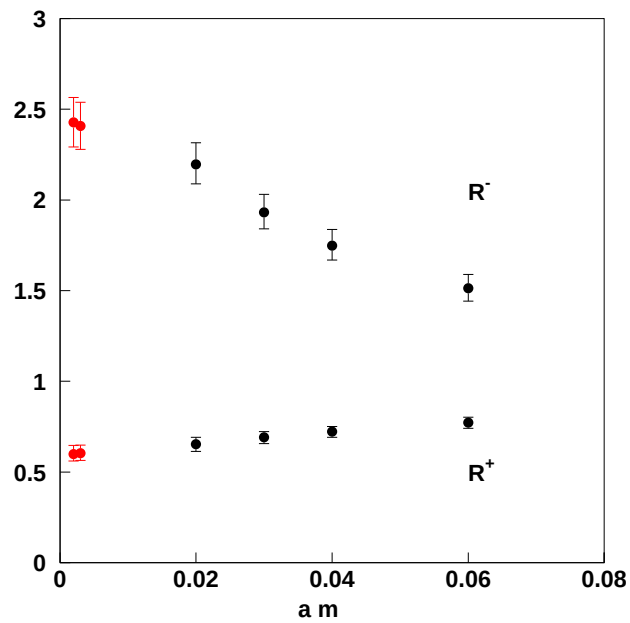
In the ϵ-regime at NLO: $\mathcal{R}^\sigma(x_0, y_0)$ independent of $x_0, y_0, |\nu|$ and any other LEC different from $g^\pm$

Tested in the quenched approximation:

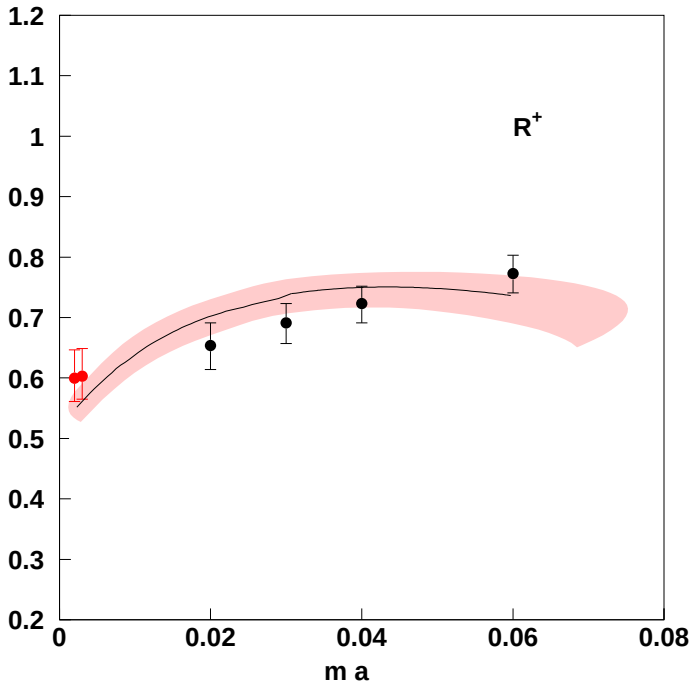
	β	L/a	T/a	$n_{\rm low}$	$L[\rm fm]$	m	# cfgs
ϵ-regime	5.8485	16	32	20	2	$m_s/40, m_s/60$	$O(800)$
p-regime	5.8485	16	32	20	2	$m_2/2 - m_s/6$	$O(200)$

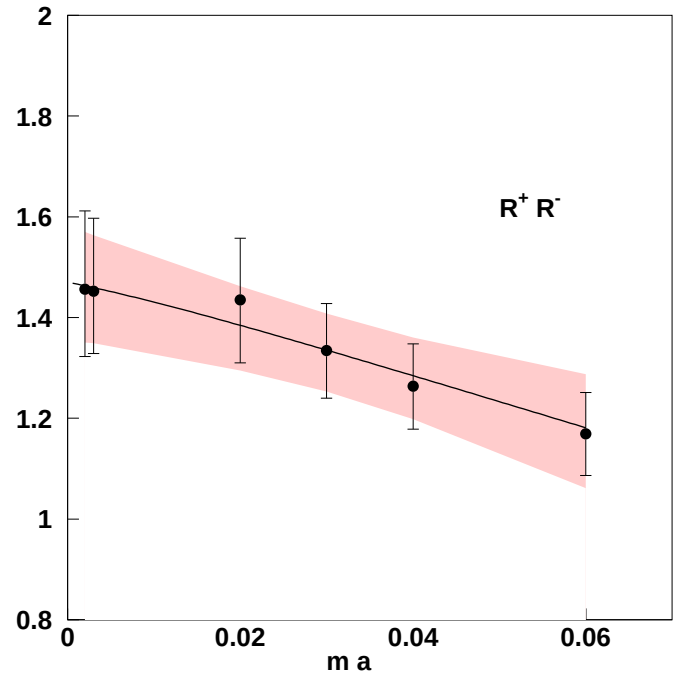
The expected features of the $R^\sigma(x_0, y_0)$ in the ϵ-regime: independence on x_0, y_0, m and ν are well reproduced by the data





$g^{\pm}$ in $SU(4)$-limit





	g^+	g^-
This work	$0.51(3)(5)(6)$	$2.6(1)(3)(3)$
"Exp"	~ 0.5	~ 10.4
Large N_c	1	1

$g^{\pm}$ from zero-mode wave functions

PH, Laine, Pena, Torró, Wennekers, Wittig in preparation

We can define e.g.

$$\bar{R}^{\sigma}_{\nu} \equiv \frac{\lim_{m\to 0}(mV)^2 \sum_{\mathrm{x,y}} \langle \partial_{x_0} P^a(x) O^{\pm}(0) \partial_{y_0} P^b(y)\rangle_{\nu}}{\lim_{m\to 0}(mV) \sum_{\mathrm{x}} \langle \partial_{x_0} P^a(x) J_{L_0}^{a}(0)\ \rangle_{\nu} \lim_{m\to 0}(mV) \sum_{\mathrm{y}} \langle \partial_{y_0} P^b(y) J_{L_0}^{b}(0)\rangle_{\nu}}$$

$$\bar{R}^{\sigma}_{\nu} = \frac{\mathcal{A}_{\nu}(x_0, y_0) \pm \tilde{\mathcal{A}}_{\nu}(x_0, y_0)}{\mathcal{B}_{\nu}(x_0)\mathcal{B}_{\nu}(y_0)}$$

$$\begin{aligned}
\mathcal{A}_{\nu}(x_0, y_0) &\equiv -\int d^3x \int d^3y \left\langle \sum_{i\in\mathcal{K}} \eta_i^{\dagger}(x) S(x,0)\gamma_{\mu} P_{-} v_i(0) \sum_{j\in\mathcal{K}} \eta_j^{\dagger}(y) S(y,0)\gamma_{\mu} P_{-} v_j(0) \right\rangle_{\nu} \\
\tilde{\mathcal{A}}_{\nu}(x_0, y_0) &\equiv \int d^3x \int d^3y \left\langle \sum_{i,j\in\mathcal{K}} \eta_j^{\dagger}(x) S(x,0)\gamma_{\mu} P_{-} v_i(0) \eta_i^{\dagger}(y) S(y,0)\gamma_{\mu} P_{-} v_j(0) \right\rangle_{\nu}
\end{aligned}$$

with $\eta_i(x) \equiv \frac{v_i(x+a\hat{0}) - v_i(x-a\vec{0})}{2}$

Matching formula:

$$g_{\pm}\bar{\mathcal{R}}_{\nu}^{\pm} = \left[k^{\pm}(M_W)\right]_{RGI} \left[\frac{Z^{\pm}}{Z_A^2}\right]_{RGI} \bar{R}_{\nu}^{\pm}$$

- The renormalization factor is the same as before !
- At LO the (q)χPT result is extremely simple:

$$\bar{\mathcal{R}}_{\nu}^{\pm} = \left(1 \mp |\nu|^{-1}\right) + \ldots$$

- NLO corrections are still quite large at 2 fm

Advantages with respect to left-current correlators:

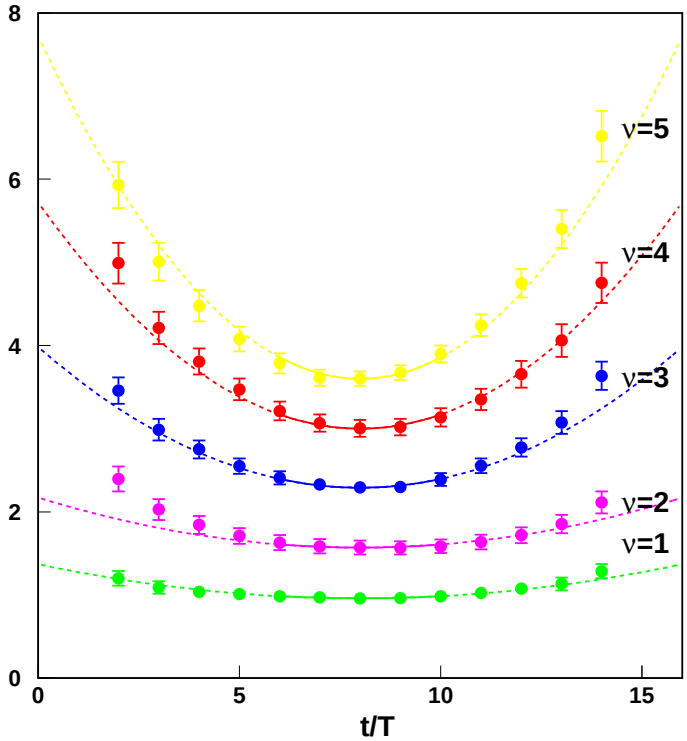
- 2 $\times|\nu|$ inversions versus 12 $+$ 2 $\times$ N_{low} to get $\sum_{x,y,\vec{0},t_0|_{x_0-t_0,y_0-t_0=\text{fixed}}}$
- Even if no low-modes are computed $\sum_{\vec{x},\vec{y},\vec{0}}$
- Completely different observable in ϵ-regime: rather different chiral corrections

Exploratory Study:

$\beta = 5.8458, V/a^4 = 16^4$, $1 \leq |\nu| \leq 5, m\Sigma V \sim \mathcal{O}(1), N_{conf} = 282$

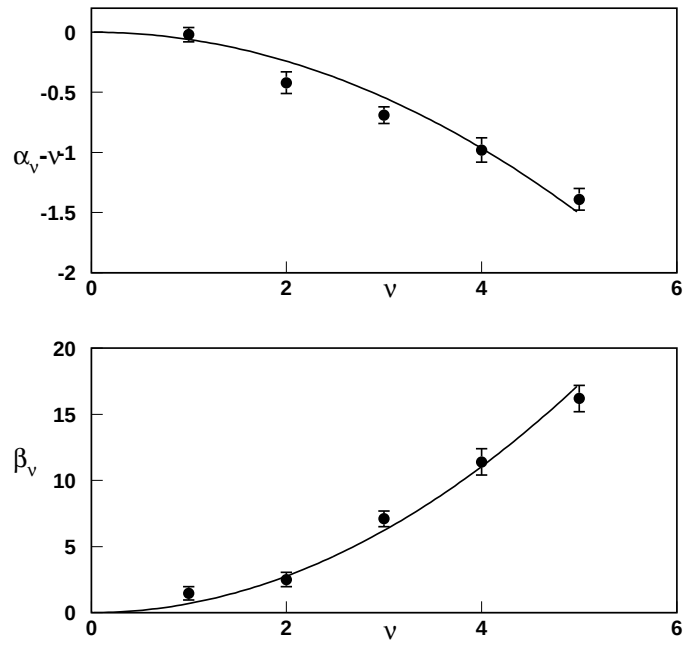
Exploratory Study: 2PT

At NLO in χPT: $\mathcal{B}_\nu(x_0) = \alpha_\nu + \beta_\nu \left(\frac{x_0}{T} - \frac{1}{2}\right)^2$



$|\nu|$ scaling of 2PT

$$\alpha_\nu = |\nu| - \frac{1}{12}\frac{T}{L}\frac{|\nu|^2}{(FL)^2} \quad \beta_\nu = \frac{T}{L}\frac{|\nu|^2}{(FL)^2}$$



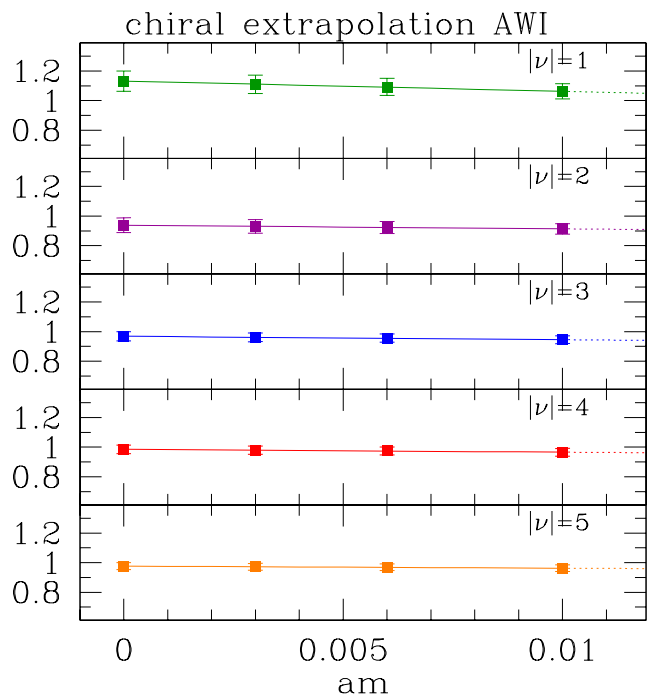
The value of F from NLO terms is in reasonable agreement with other determinations

Chiral Ward Identity from zero modes

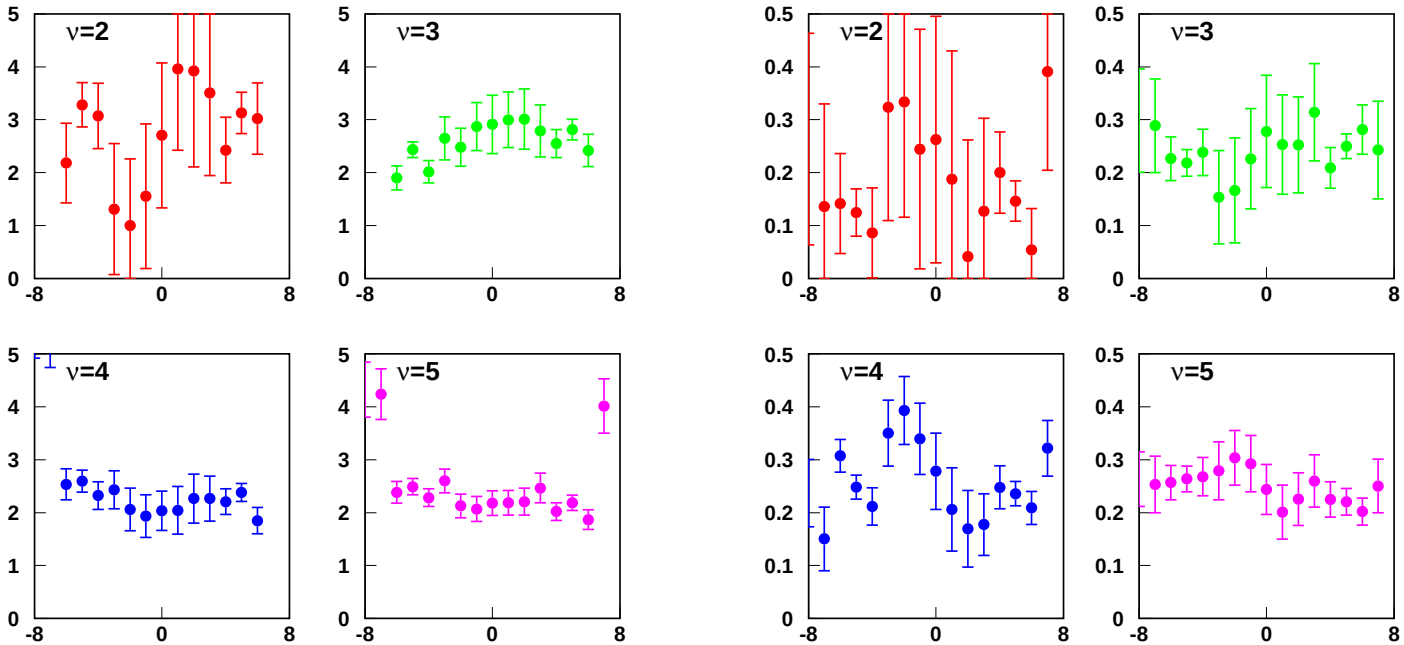
The Ward identity relates the JP and PP correlators:

$$Z_A \mathcal{B}_\nu(x_0 - y_0) = \lim_{m \to 0} m^2 V \int d^3x \langle \mathcal{P}^a(x)\, \mathcal{P}^a(y) \rangle_\nu$$

Using the standard Z_A determination



Exploratory Study: 3PT



There is a nice signal for $\bar{\mathcal{R}}_\nu^-$, less clear for $\bar{\mathcal{R}}_\nu^+$

Things seem to work as expected from the numerical point of view...but need longer time extent: work in progress !

Summary

- Fermion regularizations with exact chiral symmetry should allow us to do a precise matching to χPT

- χPT can predict not just the scaling with the quark mass, but also with L and $|\nu|$

- L, $|\nu|$ scaling in the ϵ-regime is less affected by unknown higher order LECs than the scaling with m

- Many simulations have been performed in the ϵ-regime in the quenched approximation: condensate, two and three-point functions and things seem to work as expected...

- ϵ-regime simulations in the unquenched theory are becoming feasible...