Future of Chiral Extrapolations with Domain Wall Fermions

What are the constraints on a, m, L, and particularly $m_{\rm res}$, such that one can succesfully calculate phenomenologically interesting quantities?

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The Possibilities

- GOOD: Correct continuum limit.

The Possibilities

- DARN GOOD:

The Possibilities

- DARN GOOD:
- WONDERFUL:

The Possibilities

- DARN GOOD:
- WONDERFUL:
- FANTASTIC:

Seriously . . .

- DARN GOOD: No practical barrier to calculating **some** quantities of interest (spectrum, B_K . . .) with desired precision in next 5 years.
- WONDERFUL: No practical barrier to calculating **many** quantities of interest (spectrum, B_K, ϵ'/ϵ . . .) with desired precision in next 5 years.
- FANTASTIC: No practical barrier to calculating **most** quantities of interest (spectrum, B_K, ϵ'/ϵ, condensate . . .) with desired precision in next 5 years.

Outline

- How small do a, m and $1/L$ need to be (for any fermion type)?
- Size of chiral symmetry breaking due to $m_{\mathrm{res}} \neq 0$
  - Symanzik analysis: Wilson vs. DWF?
  - Pion properties
  - Quark condensate $\bar{q}q$
  - Matrix elements of operators with power divergences: ϵ'/ϵ
  - Matrix elements without power divergences: B_K
  - Miscellany
- Conclusions

References

I am mainly drawing from work in the following articles, adding some additional observations.

- M. Bochicchio *et al.*, "Chiral symmetry on the lattice with Wilson fermions," Nucl. Phys. **B262**, 331 (1985).
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- S. R. Sharpe and R. L. Singleton, "Spontaneous flavor and parity breaking with Wilson fermions," Phys. Rev. D **58**, 074501 (1998), hep-lat/9804028.
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- Y. Shamir, "New domain-wall fermion actions," Phys. Rev. D **62**, 054513 (2000), hep-lat/0003024.
- T. Blum *et al.*, "Kaon matrix elements and CP violation from quenched lattice QCD: The 3-flavor case, Phys. Rev. D **68**, 114506 (2003), hep-lat/0110075.
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- J. Laiho and A. Soni, "Lattice extraction of $K \to \pi\pi$ amplitudes to next-to-leading order in partially quenched and in full chiral perturbation theory," Phys. Rev. D **71**, 014021 (2005), hep-lat/0306035.
- M. Golterman and Y. Shamir, "Before sailing on a domain-wall sea," Phys. Rev. D **71**, 034502 (2005), hep-lat/0411007.
- N. Christ (RBC/UKQCD), "Estimating domain wall fermion chiral symmetry breaking," PoS (LAT2005) 345.
- M. Golterman, Y. Shamir and B. Svetitsky, "Localization properties of lattice fermions with plaquette and improved gauge actions," Phys. Rev. D **72**, 034501 (2005), hep-lat/0503037.
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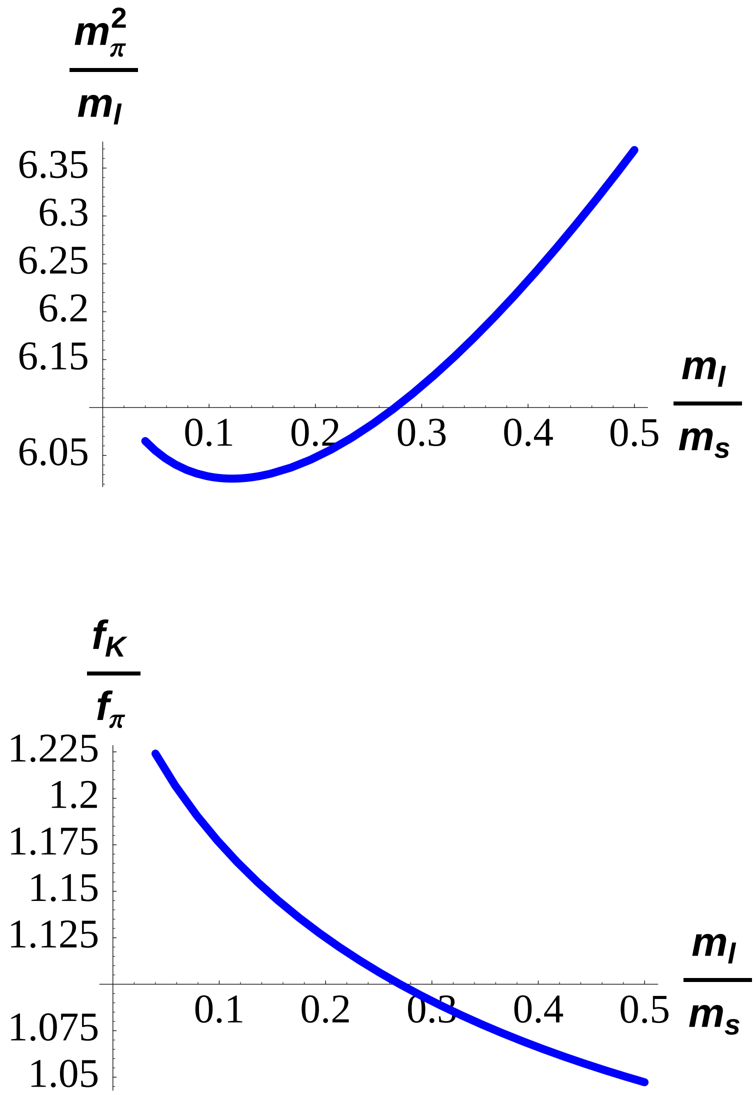
How small does a need to be?

- Dominant discretization error is $\mathcal{O}(a\Lambda)^2$
  - For $1/a = 2\,\mathrm{GeV}$, $\Lambda = 0.1 - 0.5\,\mathrm{GeV}$, $(a\Lambda)^2 = 0.003 - 0.06$
  - **Small! Are simulations with $\mathbf{a \approx 0.1\,fm}$ sufficient?**
- Same estimate holds for $O(a)$ improved Wilson fermions, but practitioners aim for $a < 0.1\,\mathrm{fm}$. Why?
  - 10-15% discretization errors at $a = 0.1\,\mathrm{fm}$ seen in quenched m_q and f_K [Garden *et al.*, hep-lat/9906013]
  - Similar effects seen for $N_f = 2$ (and Iwasaki gauge action) [Sommer *et al.*, hep-lat/0309171]
  - Large $O(a^2)$ effects in Z_A for $a \gtrsim 0.1\,\mathrm{fm}$ [Della Morte *et al.*, hep-lat/0505026]
- Do the reasons hold also for DWF? Some do, some don't.

  $\Rightarrow$ **May need smaller a for some quantities**

How small does m_q need to be?



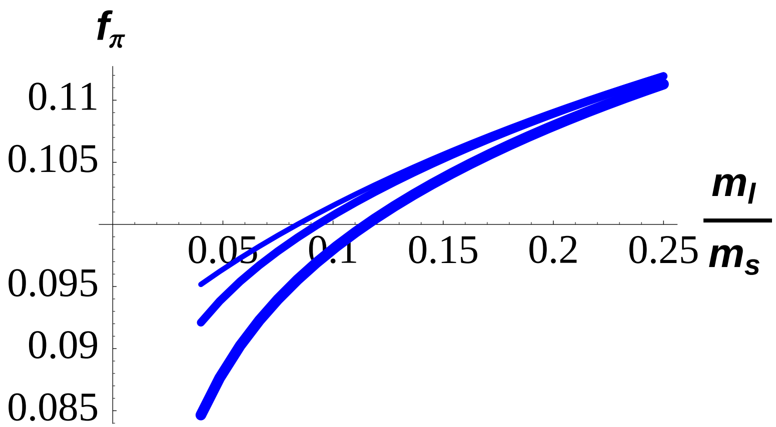
- Non-analytic terms important at small masses

  $m_s = 0.08$ GeV, $f \approx 0.09$ GeV, $L_5 = 1.45 \times 10^{-3}$, $L_8 = 10^{-3}$, $L_4 = L_6 = 0$
  [Bijnens, hep-ph/0409068]

- Must see chiral logs to have convincing extrapolations

$\Rightarrow$ **Generically, need $\mathbf{m_\ell/m_s}$ down to $\approx$ 0.1 to obtain precision results for hadronic matrix elements**

How big does L need to be?



- Finite volume effects can be substantial
  - Estimate using one-loop chiral perturbation theory
  - E.g. f_π at $a = 0.1$ fm with $L = 2.4$ fm (thick line), 3.2 fm and ∞ (thin line).

    $m_s = 0.08$ GeV,
    $f \approx 0.08$ GeV,
    $L_5 = 1.45 \times 10^{-3}$, $L_4 = 0$
- Need two-loop χPT for accurate estimate [Colangelo] , but not generally available

$\Rightarrow$ **Need L large enough that volume effects below desired precision (actual value depends on quantity)**

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  - Pion properties
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Numerical values of $m_{\rm res}$ and $a\Lambda$

- Chiral symmetry breaking parameterized by

$$m_{\rm res} \sim e^{-\alpha L_5} + \frac{\rho(0)}{L_5}$$

(**dimensionless** in this talk)

- Has a precise definition, but I will also use generically
- As move from strong to weak coupling at fixed L_5, expect $m_{\rm res}$ to decrease rapidly at first, and then asymptote
- Present $N_f = 2+1$ simulations with $L_5 = 16$ and $1/a \approx 1.6$ GeV have

$$\mathbf{m_{\rm res} = 0.003}\,; \qquad (m_{\rm res}/a \approx 5\mathrm{MeV})\,.$$

- **I use this value in subsequent estimates**, but note that $m_{\rm res}$ smaller at $1/a \approx 2.1$ GeV
- For $a\Lambda$ I use $\Lambda = 0.1 - 0.5$ GeV, so for $1/a = 1.6 - 2.16$ GeV

$$\mathbf{a\Lambda = 0.05 - 0.3}, \qquad \mathbf{1/(a\Lambda) = 3 - 20}.$$

Implications of m_{res}—Wilson fermion viewpoint

Recall classic analysis of χSB with Wilson fermions [Bochiccio *et al.*]

$$\begin{aligned} \partial_\mu A^a_\mu &= 2mP^a + aX^a \\ aX^a &\sim -2\frac{m_c}{a}P^a - (Z_A - 1)\partial_\mu A^a_\mu + O(a) \end{aligned}$$

[$\sim$ means on-shell for $p \ll 1/a$]

- Additive renormalization of quark mass: $m_q^{\text{phys}} \propto m - m_c/a$
- Axial current renormalization: $Z_A - 1 = O(g^2)$
- Need 2 conditions to determine m_c and Z_A. Can use:
  - $\langle 0|aX^a|\pi\rangle = -2(m_c/a)\langle 0|P^a|\pi\rangle - (Z_A - 1)\langle 0|\partial_\mu A^a_\mu|\pi\rangle$
  - 3-point current algebra relation

Analysis of χSB for DWF

Using "exact" axial current (and different notation!) have [Blum *et al.*]

$$\begin{aligned} \partial_\mu A^a_\mu &= 2mJ^a_5 + 2J^a_{5q} \\ J^a_{5q} &\sim \frac{m_{\rm res}}{a} J^a_5 + O(a) \end{aligned}$$

- Expect $m_{\rm res} \ll 1$ due to zero-mode decay (same holds for all terms in J^a_{5q})
- Additive renormalization of quark mass: $m_q = m + m_{\rm res}/a$
- Can determine single parameter $m_{\rm res}$ using single condition
- Standard choice is (including any $O(a)$ terms coming for the ride)

$$\frac{m_{\rm res}}{a} = \frac{\langle 0|J^a_{5q}|\pi\rangle}{\langle 0|J^a_5|\pi\rangle}$$

- Note that this implies $m_\pi^2 = 0$ when $m_q = m + m_{\rm res}(m)/a = 0$:

$$m_\pi^2 \propto \langle 0|\partial_\mu A^a_\mu|\pi\rangle = m_q \langle 0|J^a_5|\pi\rangle$$

Comparing Wilson and DWF analyses

- Why is the DWF analysis simpler? Why is the form not

$$J^a_{5q} \quad \sim \quad \frac{m_{\rm res}}{a} J^a_5 - \frac{(Z_A - 1)}{2} \partial_\mu A^a_\mu + O(a)?$$

- In fact, the $\partial_\mu A_\mu$ term is present, but highly suppressed: $Z_A - 1 \propto m_{\rm res}^2$.
- Why suppressed?
  - In perturbation theory, need additional crossing of 5th dimension to convert LR − RL to RR − LL
  - Transfer matrix argument (including zero-modes) [Christ]
  - Analog of result that $Z_A - 1 \propto r^2$ for Wilson fermions
- Effect is numerically tiny—can ignore in practice.
- Illustrates how DWF are better than "Wilson-lite"

Implications of $m_{\rm res} \neq 0$ in PGB sector

The leading $m_{\rm res}/a$ effect has been absorbed into quark mass, but what about contributions from Pauli term, suppressed by $(a\Lambda)^2$?

- Symanzik effective Lagragian for DWF (q are boundary fields):

$$\mathcal{L}_{Sym.} \sim \bar{q}(\not{D} + m)q + \frac{m'_{\rm res}}{a}\bar{q}q + ac\bar{q}(\sigma \cdot F)q + \ldots$$

  - Same form as for Wilson fermions, but here $c \sim m'_{\rm res} \ll 1$

- Match onto chiral effective theory [SS & Singleton]

$$\begin{aligned} \mathcal{L}_\chi &= \frac{f^2}{4}{\rm tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) - \frac{f^2 B}{2}{\rm tr}(M_q \Sigma + \Sigma^\dagger M_q) + \ldots \\ M_q &= m + \frac{m'_{\rm res}}{a} + ac\Lambda^2 \end{aligned}$$

  - Since $m_\pi^2 \propto M_q$, it must be that $M_q = m_q = m + m_{\rm res}/a$ and so

$$m_{\rm res} = m'_{\rm res} + ca^2\Lambda^2$$

- **Conclusion: No $\mathbf{m_{\rm res} a \Lambda^2}$ term in leading order $\mathcal{L}_\chi$ if use standard $\mathbf{m_{\rm res}}$**

Implications for PGBs (continued)

This "trick" does not work for higher order terms:

- PGB matrix elements of $\bar{q}\sigma \cdot Fq$ and $\bar{q}q$ not proportional at higher order
- Extra terms obtained by replacement $m_q \to m_{\rm res} a\Lambda^2$

$$\frac{m_\pi^2}{m_q} \sim f_\pi \sim \text{const.} \left[1 + \mathcal{O}(m_q/\Lambda) + \mathcal{O}(m_{\rm res} a\Lambda) + \mathcal{O}(a^2\Lambda^2) + \ldots\right]$$

- Here $m_{\rm res}$ indicates order of magnitude—dependence on L_5 may differ
- **Numerically tiny and subdominant to $\mathbf{a^2\Lambda^2}$ term:**

$$m_{\rm res} a\Lambda \approx 0.003 \left(\frac{0.1 - 0.5\ \text{GeV}}{1.6\ \text{GeV}}\right) \lesssim 10^{-3} \ll (a\Lambda)^2 \approx 0.004 - 0.1$$

- **Similar $\mathbf{O(a)}$ term present in all hadronic quantities**

Mass dependence of $m_{\rm res}$

- Results show small linear dependence: how large parametrically?
- Recall Symanzik expansion for "mid-point pseudoscalar density"

$$J^a_{5q} \quad \sim \quad \frac{m'_{\rm res}}{a} J^a_5 (1 + a^2 m^2) + m^2_{\rm res} \partial_\mu A^a_\mu + a c \bar{q} \sigma \cdot F \gamma_5 T^a q + \dots$$

  [$m'_{\rm res}$ and $c \sim m'_{\rm res}$ same as above]
- Dominant linear effect is from different m dependence of $\langle 0 | J^a_5 | \pi \rangle$ and $\langle 0 | \bar{q} \sigma \cdot F \gamma_5 q | \pi \rangle$
- It is **quadratic** in a

$$m_{\rm res}(m_q) = m_{\rm res}(m_q = 0) \left[1 + \mathcal{O}(m_q a^2 \Lambda) \right]$$

- **Is this consistent with observed size?**
- **Puzzle: why is valence quark dependence of $\mathbf{m}_{\rm res}$ stronger?** (as seen in talk by [M. Lin])

Enhanced m_{res} effects: condensate

- For the χSB induced by m_{res} to be problematic, must be enhanced
  - This can be due either to power divergences or mixing with operators with less suppressed chiral behavior.
- Most extreme case is quark condensate:
  - Symanzik expansion of scalar

$$(\bar{q}q)\Big|_{\text{DWF}} \sim (\bar{q}q)\Big|_{\text{cont.}} + \frac{m + x m_{\text{res}}/a}{a^2} + \dots$$

  - $x = \mathcal{O}(1)$ but $x \neq 1$ because term arises from UV momenta where cannot use Symanzik action
  - Thus do not remove divergence by $m_q = m + m_{\text{res}}/a \to 0$ [Blum *et al.*]

$$\lim_{m_q \to 0} \lim_{L \to \infty} \langle \bar{q}q \rangle_{\text{DWF}} = \langle \bar{q}q \rangle_{\text{cont}} + (x-1)\frac{m_{\text{res}}}{a^3} + \dots$$

- Relative correction is large

$$\frac{\delta \langle \bar{q}q \rangle}{\langle \bar{q}q \rangle} \sim \frac{m_{\text{res}}}{a^3 \Lambda^3} \sim \frac{3 \times 10^{-3}}{(0.1)^3} \sim O(1)$$

$\Rightarrow$ **Cannot calculate physical condensate directly** (although indirect methods can work)

Enhanced $m_{\rm res}$ effects: ϵ'/ϵ

- To test whether the SM predicts the measured ϵ' need $K \to \pi\pi$ matrix elements (ME) of operators such as

$$\mathcal{O}_6 = \bar{s}\gamma_\mu P_L d \sum_q \bar{q}\gamma_\mu P_R q$$

  with a precision of 10-30%

- Direct calculation of physical matrix elements challenging; avoid by using χPT
  - LO [Bernard *et al.*] : use $K \to 0$ and $K \to \pi$ with $m_K = m_\pi$
  - NLO [Laihi & Soni] : use $K \to 0$, $K \to \pi$ with $m_K \neq m_\pi$, $K \to \bar{K}$, and unphysical $K \to \pi\pi$ with pions at rest
- Part of complication is to account for mixing with the lower-dimension operator

$$\mathcal{O} = \frac{1}{a^2}\left[(m_d - m_s)\bar{s}\gamma_5 d + (m_s + m_d)\bar{s}d\right]$$

Impact on ϵ'/ϵ (continued)

- With DWF get additional lower-dimension operators to subtract

$$\frac{m_{\rm res}}{a^3}\bar{s}d\,,\ \ \frac{m_{\rm res}}{a}\bar{s}\sigma\cdot Fd\,,\ \ \frac{m_{\rm res}(m_s\pm m_d)^2}{a}\bar{s}d\,,\ \ \frac{m_{\rm res}(m_s^2-m_d^2)}{a}\bar{s}\gamma_5 d$$

- How do they impact the calculation?
- $m_{\rm res}\bar{s}d/a^3$ removed by RBC "slope" method, while $m_{\rm res}\bar{s}\sigma\cdot Fd/a$ leads to relative correction no larger than omitted NNLO terms:

$$\frac{\delta ME}{ME}\sim\frac{m_{\rm res}}{a\Lambda}\sim\frac{3\times 10^{-3}}{0.06-0.3}\sim 0.01-0.05$$

- Contributions of other operators suppressed by further $m_s/\Lambda\approx 1/4$
- **Conclusion: Present parameters probably adequate, though worth investigating methods for subtracting additional operator**
- See talk by [Christ] for an additional operator

Though need to then do some extra work to implement Laiho-Soni.

Implications of $m_{\rm res}$ for B_K

- Want to calculate B_K to at least 5% precision [Talk by Soni]
- No power divergent mixing, so effects of $m_{\rm res}$ suppressed by $a\Lambda$ as for spectral quantities
- However, L-L operator can mix with chirally unsuppressed L-R operator: does this enhance the usual $m_{\rm res} a\Lambda$ corrections?
- No! Mixing comes at cost of m_q and gain of $1/m_q$. Net result is

$$\frac{\delta B_K}{B_K} \sim m_{\rm res} \times (a\Lambda) \lesssim 10^{-3}\,.$$

- **Conclusion: present parameters are adequate for precision calculation of $\mathbf{B_K}$**

Lorentz violations at $O(a^2)$

- Symanzik action contains $a^2 \sum_\mu \bar{q} D_\mu^3 \gamma_\mu q$
- In pion sector, "Lorentz"-violating effects proportional to $a^2 \sum_\mu p_\mu^4$ and thus very small
- For masses of other hadrons Lorentz-violation occurs at lower order: e.g. m_N can contain a $(a^2 \vec{s} \cdot \vec{p})^2$ term
- **Interesting diagnostic of size of $\mathbf{a^2}$ terms that is worthwhile studying**

Pion EM splitting

[Gupta, Kilcup & SS, 1984] —based on suggestion by David Kaplan

- Direct calculation of EM splittings has begun [Talk by Doi]
- At leading order in χPT [Das *et al.*, 1967]

$$m_{\pi^+}^2 - m_{\pi^0}^2 = \pm \frac{2e^2}{f^2} \int d^4x \frac{1}{4\pi^2 x^2} \langle \bar{u}_L \gamma_\mu d_L(x)\ \bar{d}_R \gamma_\mu u_R(0) \rangle$$

  where RHS should be extrapolated to chiral limit

- Requires chiral symmetry, and with DWF expect

$$\begin{aligned} \frac{m_{\pi^+}^2 - m_{\pi^-}^2}{2e^2/f^2} &\sim f^4 + \frac{m_{\rm res}^2}{a^4} \\ &\sim f^4 \left[1 + \frac{m_{\rm res}^2}{(af)^4}\right] \\ &\sim f^4 \left[1 + \frac{10^{-5}}{1.5 \times 10^{-5}}\right] \end{aligned}$$

- **Probably not practical, but perhaps worth a more detailed look**

Conclusions?

D, W or F?

My Conclusion:

- WONDERFUL: No practical barrier to calculating **many** quantities of interest (spectrum, B_K, $\epsilon'/\epsilon \ldots$) with desired precision in next 5 years.
- Need $m_\ell/m_s \approx 0.1$, $L \approx 4\,\mathrm{fm}$, $a \approx 0.06\,\mathrm{fm}$ and $m_{\rm res} \approx 10^{-3}$, which are attainable parameters
- Need to include heavy quarks (particularly b)
- Need to extend our repetoire [Soni's talk]
- **Next meeting should be very interesting!**

THANKS TO AMARJIT and TOM!