Non-perturbative renormalization of 4 fermi operator with Schrödinger functional scheme

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Purpose of this talk

- $\widehat{Z}_{B_K}$: Bare $B_K^{(0)}$ on lattice with DWF
  $\Rightarrow$ Renormalization group invariant (RGI) $\widehat{B}_K$

- Three steps for RGI $\widehat{B}_K$ (SF scheme)

1. From lattice to renormalized: $B_K^{(0)} \Rightarrow B_K^{(\mathrm{SF})}(a\mu)$
   - lattice artifact: $\mathcal{O}(a\mu) \Rightarrow a\mu \ll 1$
2. Non-pert. RG evolution: $B_K^{(\mathrm{SF})}(a\mu_{\mathrm{min}}) \Rightarrow B_K^{(\mathrm{SF})}(a\mu_{\mathrm{max}})$
3. Pert. transformation to RGI $B_K^{(\mathrm{SF})}(a\mu_{\mathrm{max}}) \Rightarrow \widehat{B}_K$

- 2. 3. steps are done by Alpha-collab. for O_{VA+AV}
- We need to evaluate 1. for DWF with O_{VA+AV}

Renormalization of B_K

$$B_K = \frac{\left\langle \overline{K}^0 \left| \left(\overline{s}\gamma_\mu^L d\right)\left(\overline{s}\gamma_\mu^L d\right) \right| K^0 \right\rangle}{\frac{8}{3}\left\langle \overline{K}^0 \left| \overline{s}\gamma_\mu\gamma_5 d \right| 0 \right\rangle \left\langle 0 \left| \overline{s}\gamma_\mu\gamma_5 d \right| K^0 \right\rangle}$$

- 4 fermi operator

$$\left(\overline{\psi}_1\gamma_\mu^L\psi_2\right)\left(\overline{\psi}_3\gamma_\mu^L\psi_4\right) = O_{VV+AA} - O_{VA+AV}$$
$$O_{VV+AA} = \left(\overline{\psi}_1\gamma_\mu\psi_2\right)\left(\overline{\psi}_3\gamma_\mu\psi_4\right) + \left(\overline{\psi}_1\gamma_\mu\gamma_5\psi_2\right)\left(\overline{\psi}_3\gamma_\mu\gamma_5\psi_4\right)$$
$$O_{VA+AV} = \left(\overline{\psi}_1\gamma_\mu\psi_2\right)\left(\overline{\psi}_3\gamma_\mu\gamma_5\psi_4\right) + \left(\overline{\psi}_1\gamma_\mu\gamma_5\psi_2\right)\left(\overline{\psi}_3\gamma_\mu\psi_4\right)$$

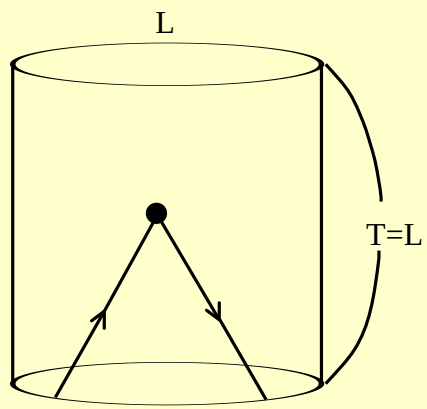
- Renormalization factor for B_K

$$Z_{B_K} = \frac{Z_{VV+AA}(g_0, a\mu)}{Z_A^2(g_0)}$$

- We believe in chiral symmetry of DWF

$$Z_{VV+AA} = Z_{VA+AV}, \quad Z_V = Z_A \text{ (CP} - \text{PACS)}$$
$$Z_{B_K} = \frac{Z_{VA+AV}(g_0, a\mu)}{Z_V^2(g_0)}$$

Schrödinger functional scheme (Lüscher et al)

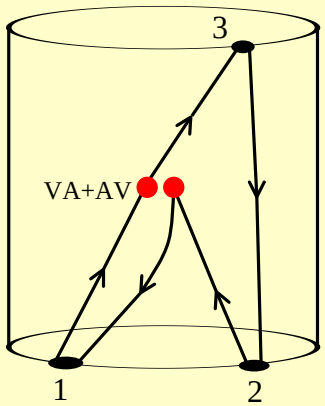


- Finite box $L^3 \times T \sim L^4$
- Dirichlet BC in time

$$U_k|_{x_0=0} = 1, \quad U_k|_{x_0=T} = 1$$
$$P_+\psi|_{x_0=0} = 0, \quad P_-\psi|_{x_0=T} = 0$$
$$P_\pm = 1 \pm \gamma_0$$

- Renormalization scale $\sim 1/L$
- Finite mass gap $\sim 1/T$ (Sint)
  - mass independent scheme ($m = 0$) exactly
- Step scaling function (SSF)
  - non-pert. evolution of renormalization scale

Renormalization of O_{VA+AV} (Alpha-collab.)



- **Four point function**

$$F(x_0) = \frac{1}{L^3}\langle \mathcal{O}_1 \mathcal{O}_2 O_{VA+AV}(x) \mathcal{O}'_3 \rangle$$

$$\mathcal{O}[\Gamma] = a^6 \sum_{\vec{x},\vec{y}} \overline{\zeta}(\vec{x})\Gamma\zeta(\vec{y})$$

- **Boundary fields**

$$\zeta(\vec{x}) \sim P_-\psi(x)|_{x_0=0} \qquad P_+\psi|_{x_0=0} = 0$$

$$\Gamma = \gamma_5,\ \gamma_k$$

- **Boundary operator is renormalized with wave function renormalization factor (Sint)**

$$\mathcal{O}^{(0)} = Z_\zeta^2 \mathcal{O}^{(R)}$$

- Z_ζ is cancelled dividing by two point function

$$f_1 = -\frac{1}{2L^6}\langle \mathcal{O}'[\gamma_5]\mathcal{O}[\gamma_5]\rangle$$

- Definition of renormalization factor

$$h(x_0) = \frac{F(x_0)}{f_1^{3/2}}$$

$$Z_{VA+AV}(g_0, a\mu)h^{(0)}(\frac{T}{2}; g_0) = h^{(\mathrm{tree})}(\frac{T}{2}; g_0 = 0)$$

- Setting external mom. $\Leftrightarrow$ corr. function with an extent
- Drop external legs $\Leftrightarrow$ divide by f_1

Orbifolding construction of SF

- Dirichlet BC $(1+\gamma_0)\psi(0)=0$ breaks chiral symmetry
  $\Leftarrow$ consistent with DWF chiral symmetry (bad!)

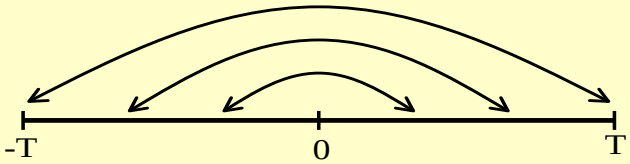
- Field theory with DBC $\Leftrightarrow$ orbifolded field theory
  - Easily found in the continuum
  - Easy to apply to DWF

  $\Rightarrow$ chiral symmetry breaking by projection

- Double the temporal direction and fold by projection
  - Anti-periodic BC in $2T$
  - gauge fields are copied

    $U_k(x_0) = U_k(-x_0)$

    $U_0(x_0) = U_0^\dagger(-x_0-1)$
  - orbifolding: $x_0 \leftrightarrow -x_0$



Numerical simulation

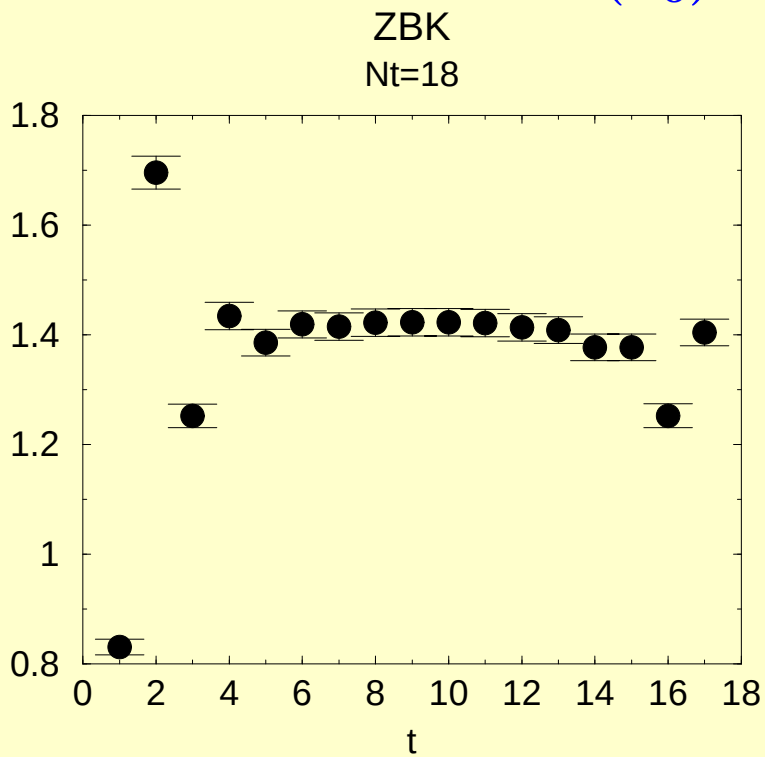
- Quenched simulation data for B_K with Iwasaki action and DWF ($N_s = 16$)
  at $\beta = 2.6$, $\beta = 2.9$, $\beta = 3.2$
  corresponding to $a^{-1} = 2$, $a^{-1} = 3$, $a^{-1} = 4$ GeV
- We need renormalization factor $Z_{VA+AV}(a\mu)$
  at $\mu_{\rm min} = 1/2L_{\rm max} = 1/(1.436 r_0) \sim 275$ MeV
  $a\mu_{\rm min} = 0.17 \ - \ 0.05$
- Need to fine tune β such that $2aN_T = 1.436 r_0$

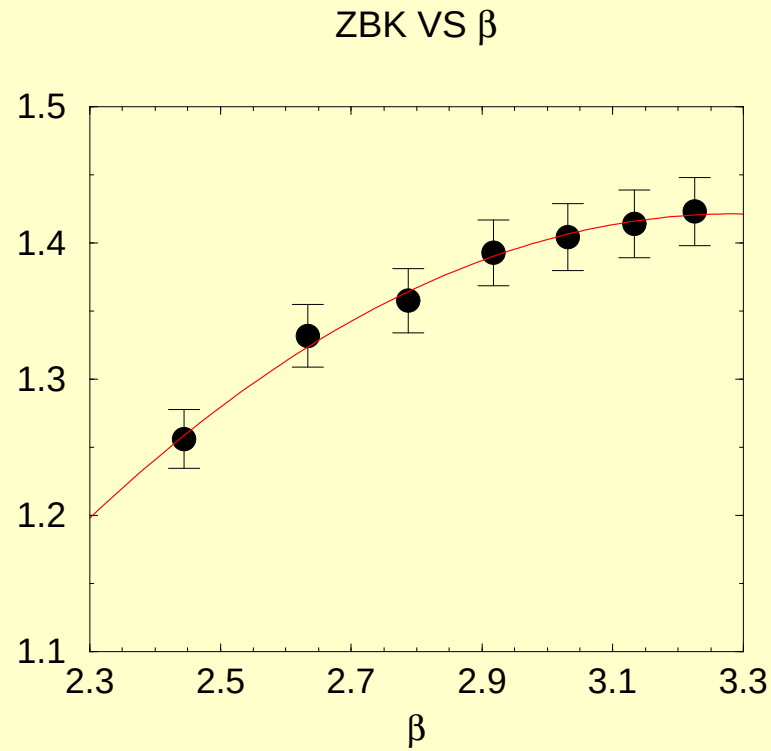
N_T	6	8	10	12	14	16	18
β	2.44	2.63	2.79	2.92	3.03	3.13	3.23

- $\mu_{\rm max} = 2^7 \mu_{\rm min} \sim 35.2$ GeV
- Values at $\beta = 2.6$, $\beta = 2.9$, $\beta = 3.2$ by interpolation

Numerical results

- **4 point function** $h(x_0)$





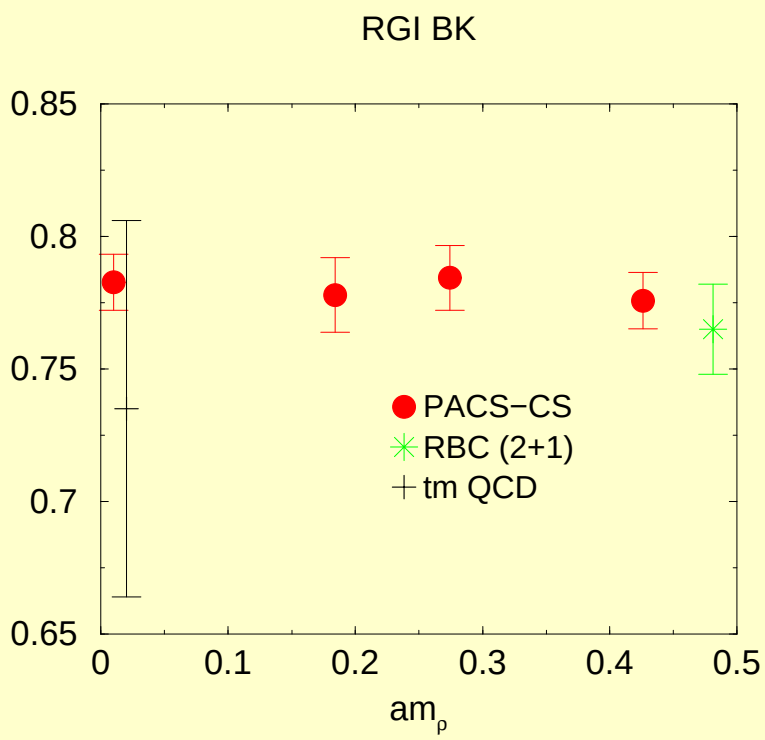
- **Interpolation of** $\widehat{Z}_{B_K}$ $\Uparrow$

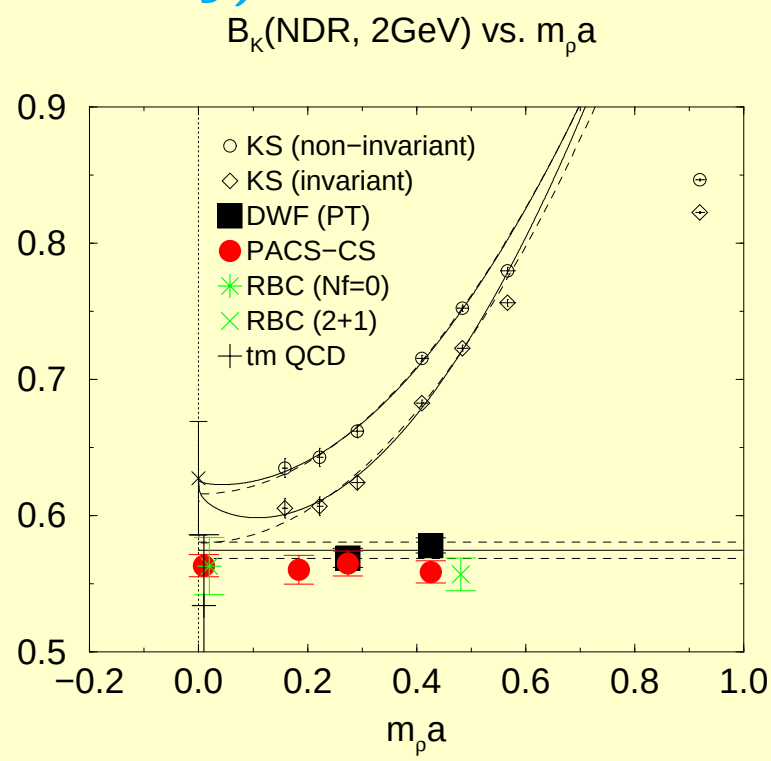
$$\widehat{Z}_{B_K}(\beta) = a + b(\beta - 3) + c(\beta - 3)^2$$

$$\chi^2/\mathrm{dof} = 0.07$$

Renormalized B_K

- **RGI** $\widehat{B}_K = 0.783(11)$ **(preliminary)**





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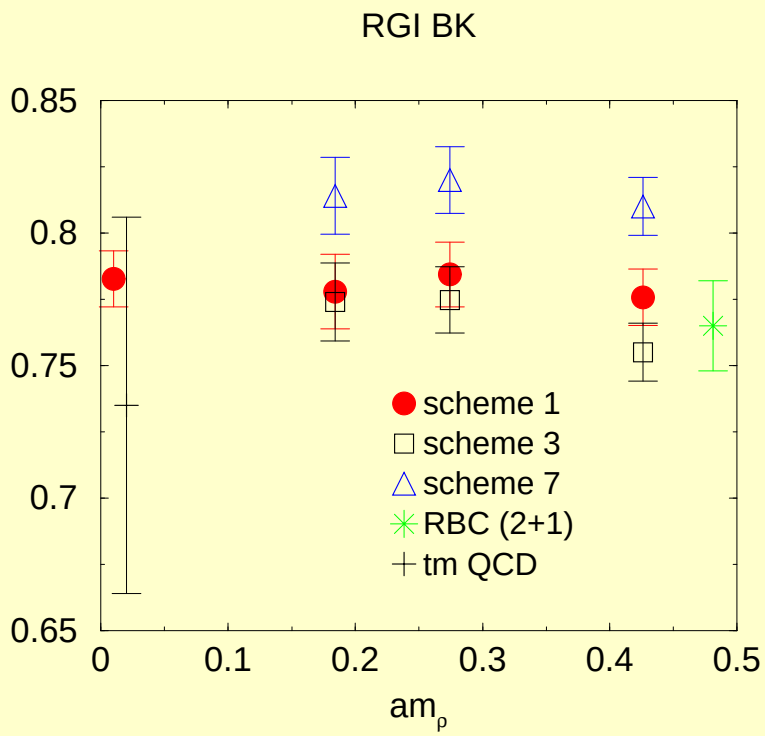
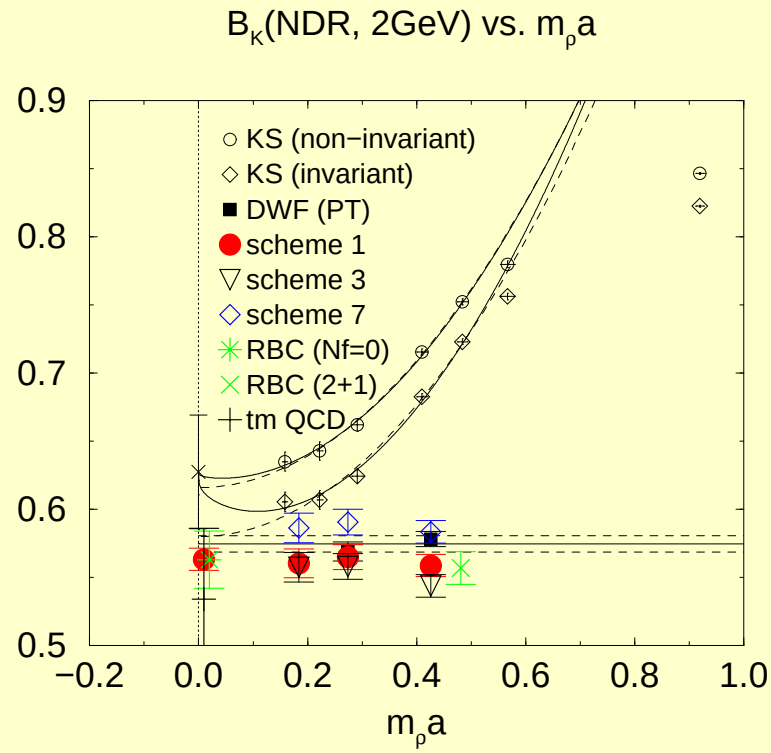
- **Renormalized** B_K **in** $\overline{\mathrm{MS}}$ **NDR scheme at** $\mu = 2$ **GeV**

  $B_K^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 0.5633(81)$ **(preliminary)**

Scheme dependence?

- **RGI** $\widehat{B}_K = 0.783(11)$ **(preliminary)**





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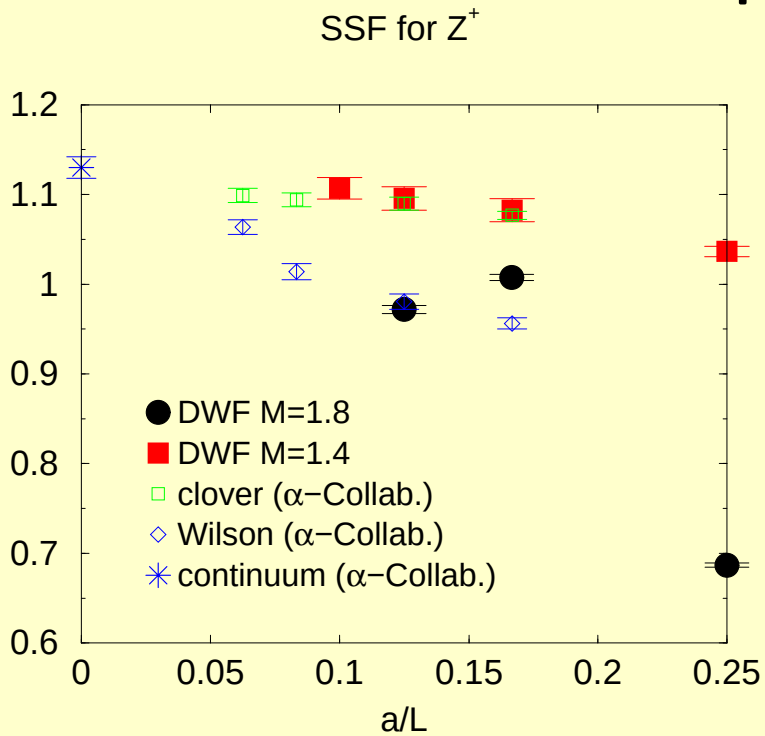
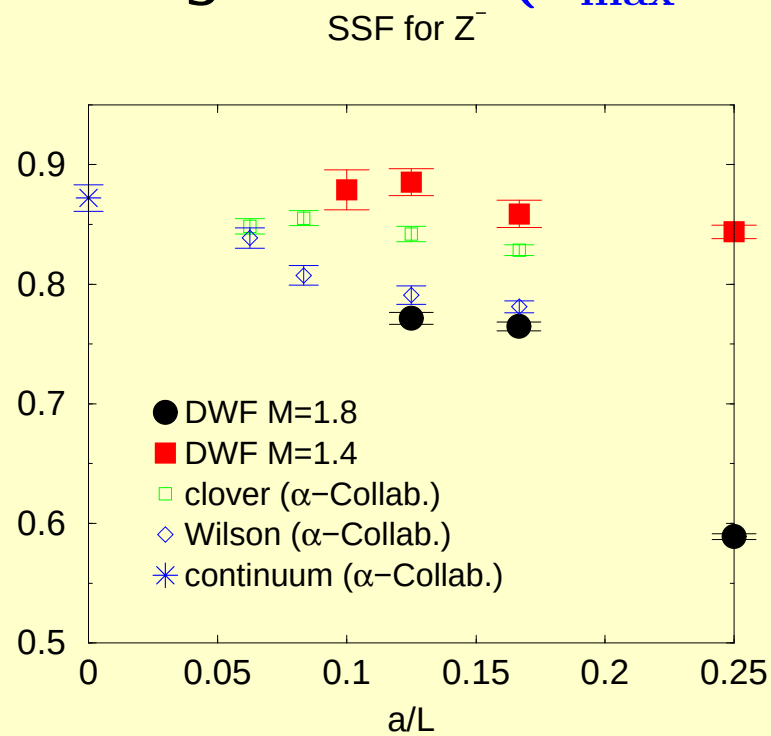
Step scaling function

- **Continuum limit of step scaling function** $(L_{\max} \to 2L_{\max})$



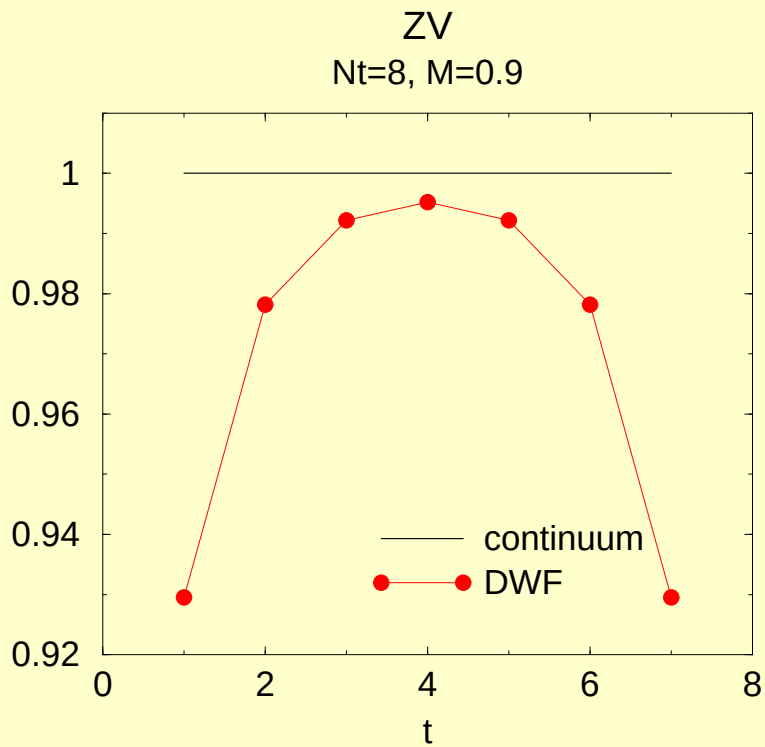


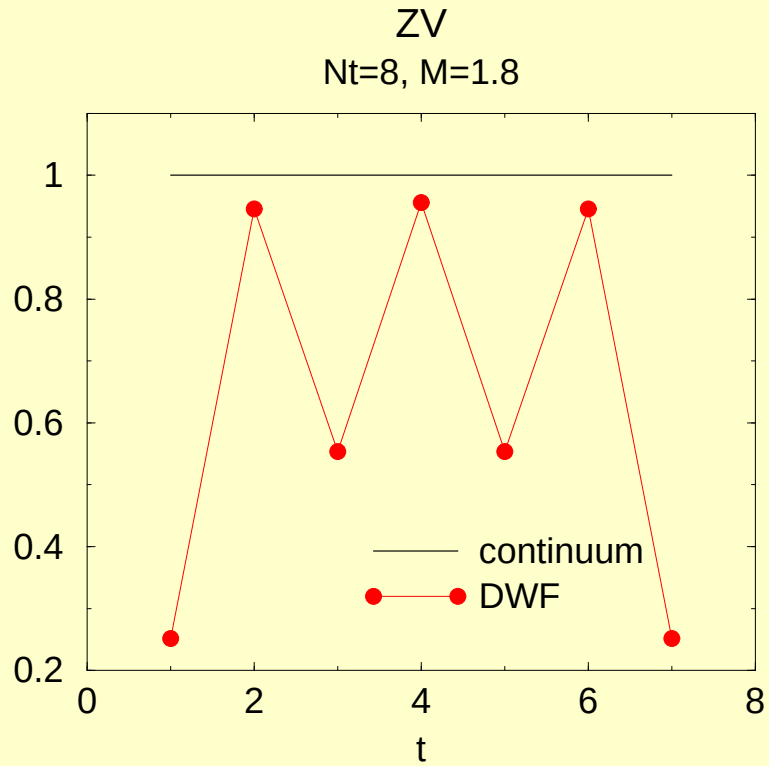
For Z^{+}_{VA+AV} For Z^{-}_{VA+AV}

$$O^{\pm}_{\Gamma\Gamma'} = (\overline{\psi}_1\Gamma\psi_2)\,(\overline{\psi}_3\Gamma'\psi_4) \pm (\overline{\psi}_1\Gamma\psi_4)\,(\overline{\psi}_3\Gamma'\psi_2)$$

Tree level Green's function for Z_V

- For $N_T = 8$



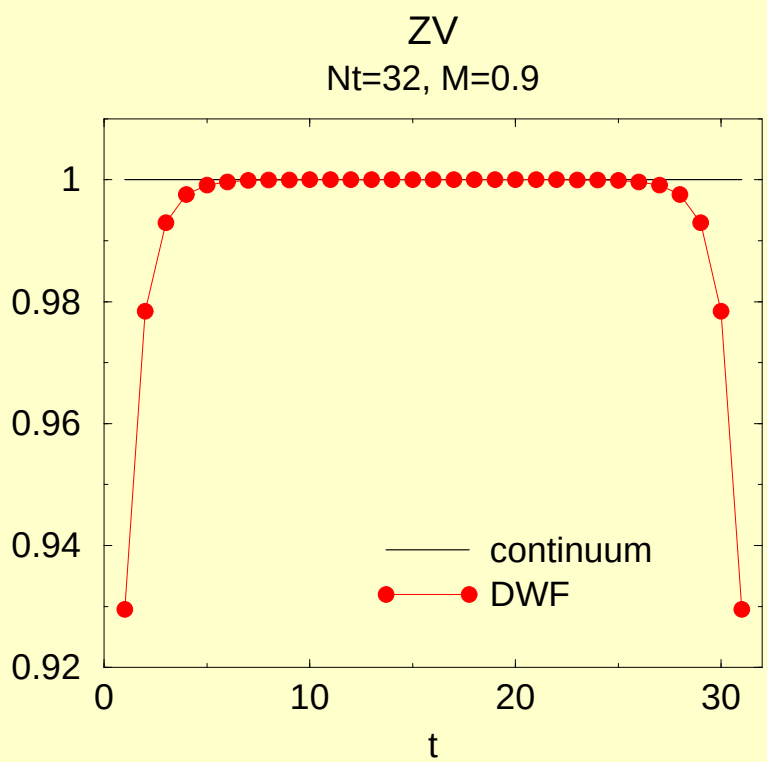


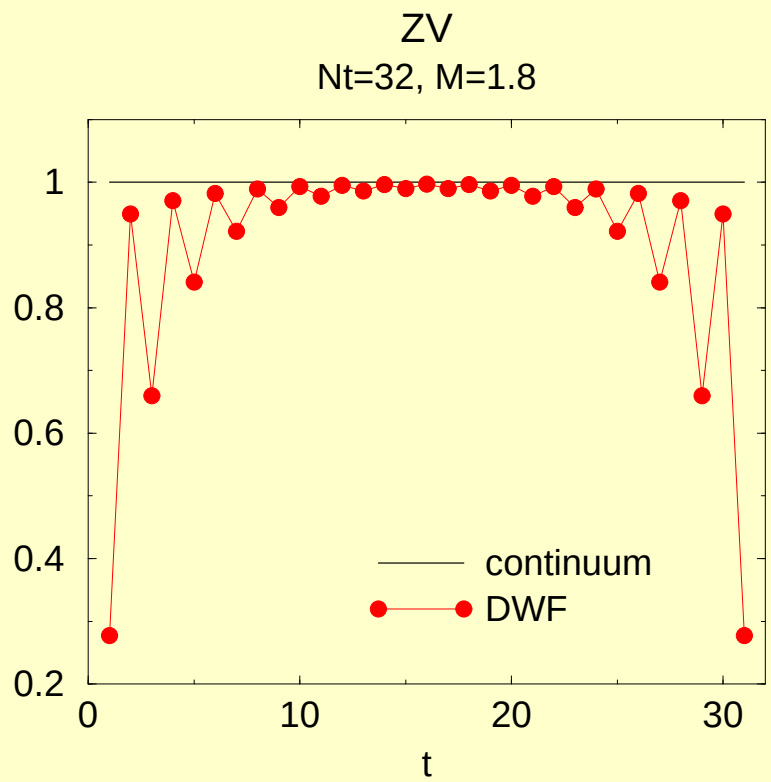
For $M = 0.9$

For $M = 1.8$

Tree level Green's function for Z_V

- For $N_T = 32$





For $M = 0.9$

For $M = 1.8$

Conclusion

- $\widehat{Z}_{B_K}$: Bare $B_K^{(0)}$ on lattice with DWF
  $\Rightarrow$ Renormalization group invariant (RGI) $\widehat{B}_K$

- Schrödinger functional scheme for DWF
  with orbifolding construction

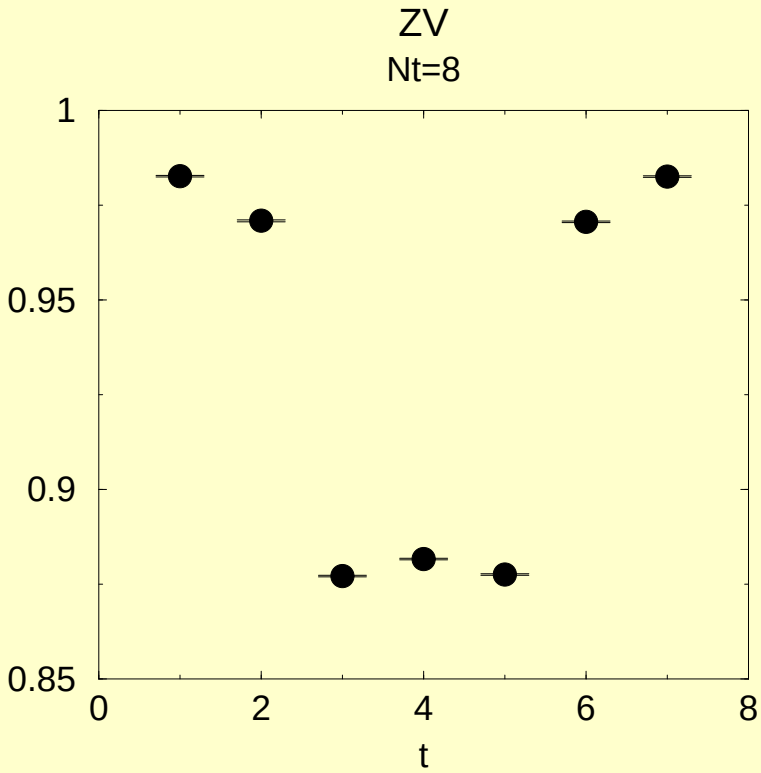
- RGI $\widehat{B}_K = 0.783(11)$ (preliminary)

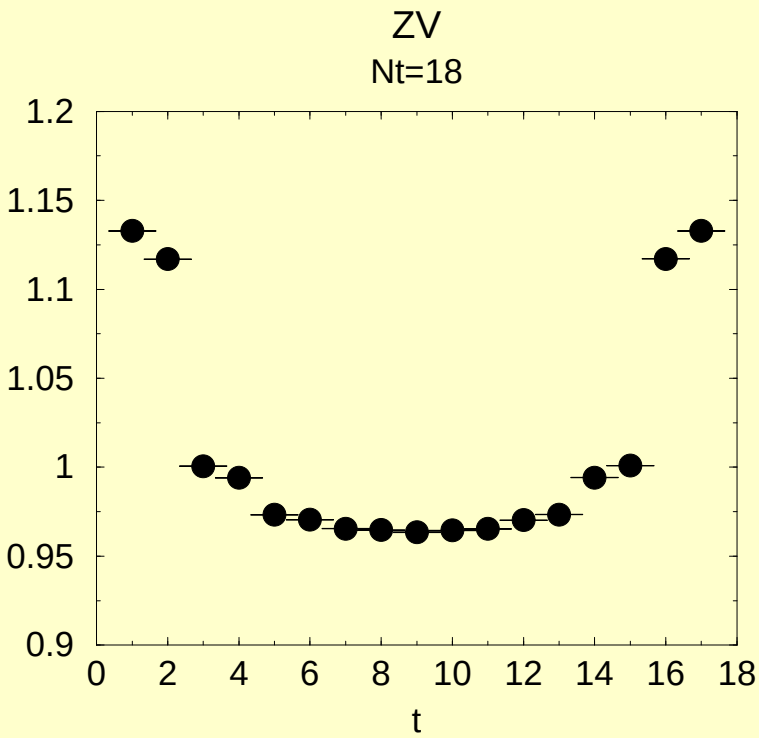
- $\overline{\mathrm{MS}}$ $B_K^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 0.5633(81)$ (preliminary)

- $\mathcal{O}(a)$ behaviour depends on domain-wall height M
  - M should be tuned such that $M - \delta M_{\mathrm{tadpole}} \sim 1$

Figures

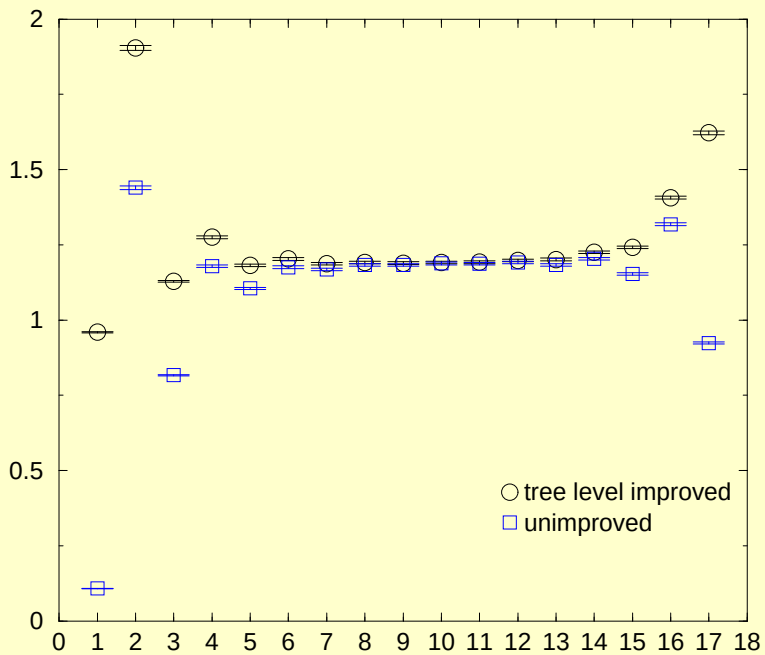
- **Three point function for Z_V**

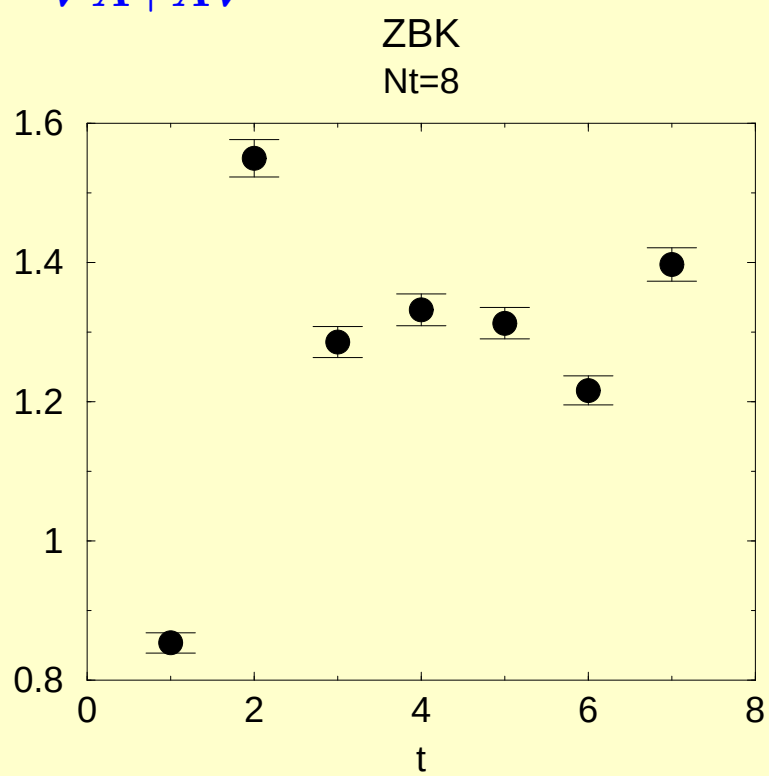




Figures

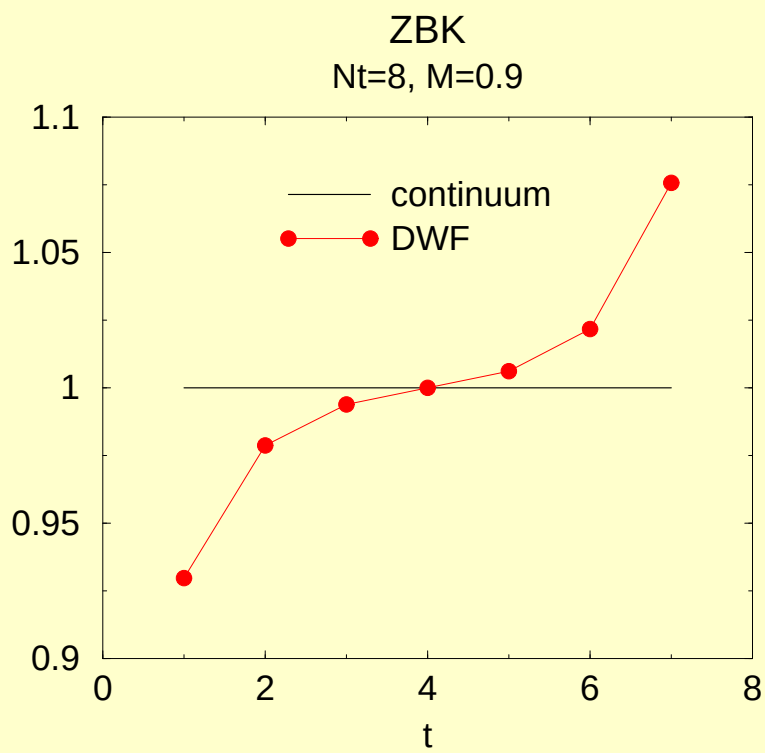
- Correlation function for Z^{+}_{VA+AV}



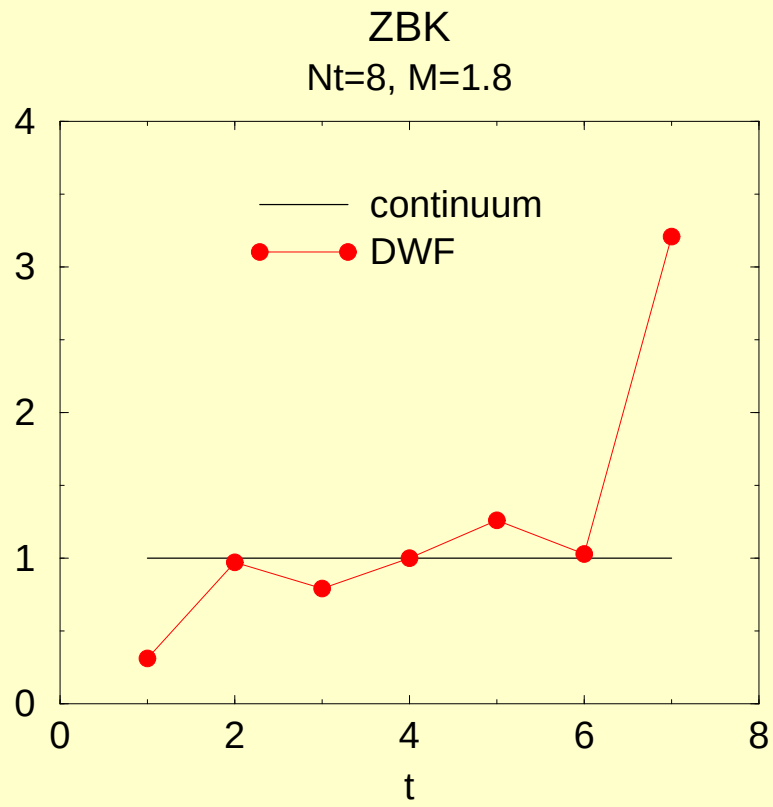


Tree level Green's function for Z_{B_K}

- For $N_T = 8$



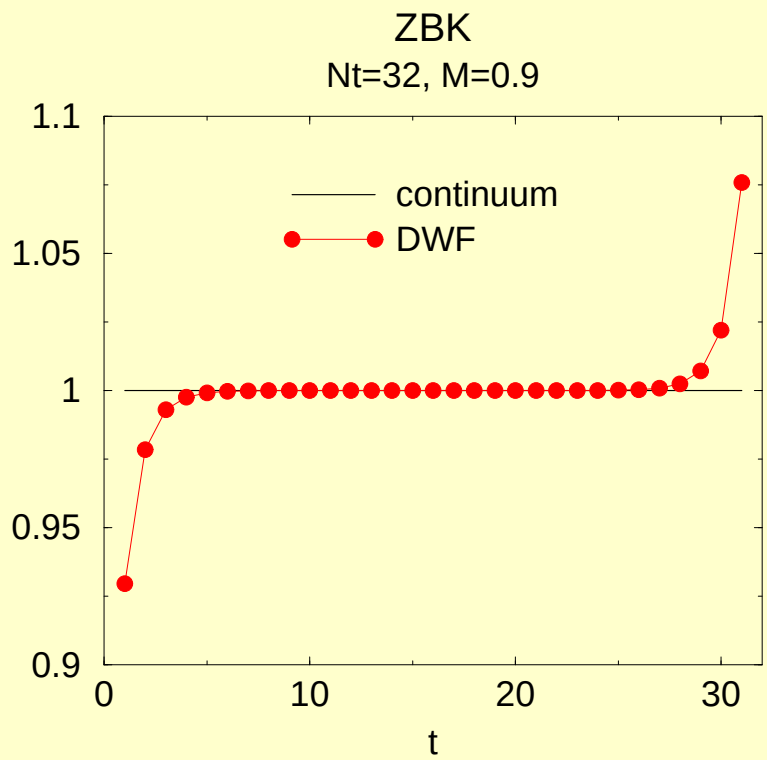
For $M = 0.9$



For $M = 1.8$

Tree level Green's function for Z_{B_K}

- For $N_T = 32$



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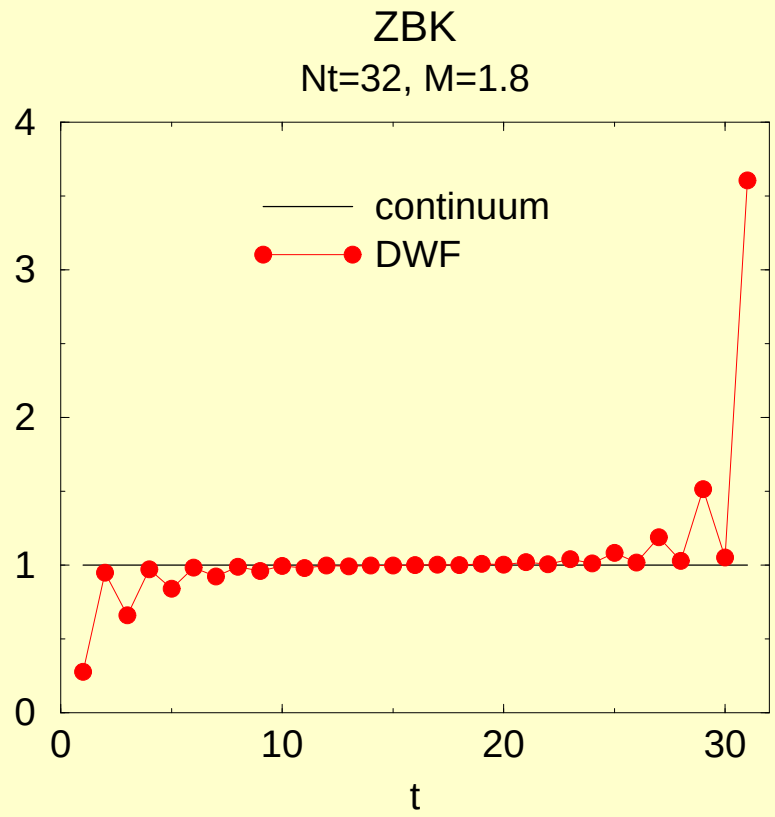


For $M = 1.8$

Problem with naive formulation

zero mode problem

- Wilson Dirac operator in DWF

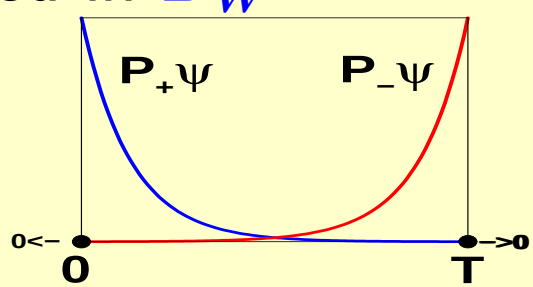
$$
\begin{aligned}
D_W &= -\frac{1-\gamma_0}{2}\delta_{m_0,n_0+1} - \frac{1+\gamma_0}{2}\delta_{m_0,n_0-1} + D_W^{(3)} + (1-M) \\
&= \begin{pmatrix} D_W^{(3)} + M + 1 & -P_- & \\ -P_+ & D_W^{(3)} + M + 1 & -P_- \\ & -P_+ & D_W^{(3)} + M + 1 \end{pmatrix}
\end{aligned}
$$

- SF BC: $P_+\psi|_{x_0=0} = 0$, $P_-\psi|_{x_0=T} = 0$
- Signature of r, M fixed to eliminate doubler

  (for $r = 1$; $-1 < 1 - M < 1$)

$\Rightarrow$ zero mode solution is allowed in D_W

$$
\begin{aligned}
\psi &= P_-(1-M)^{x_0} \\
&\quad + P_+(1-M)^{(T-x_0)}
\end{aligned}
$$



Orbifolding projection for DWF

- time reversal symmetry for domain-wall fermion

  $\psi \to i\gamma_0\gamma_5 PR\psi, \quad \overline{\psi} \to \overline{\psi} i\gamma_0\gamma_5 PR,$

  R: time reversal $R\psi(x_0) = \psi(-x_0)$

  P: parity flip in 5-th direction

- chiral symmetry (Furman-Shamir)

$$\psi \to -iQ\psi, \quad \overline{\psi} \to \overline{\psi} iQ, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  Q : chiral charge in 5-th direction

- anti-periodicity $\psi(x_0 + 2T) = -\psi(x_0)$

  $R\psi(0) = \psi(0), \quad R\psi(T) = -\psi(T)$: fixed points

- Orbifolding projection

$$(1 - A)\psi(x) = 0, \quad \overline{\psi}(x)(1 - A) = 0, \quad A = \gamma_0\gamma_5 PQR$$

- **Orbifolding projection for physical quark**

$$q = \left(P_L \delta_{s,1} + P_R \delta_{s,N_5}\right)\psi, \quad \overline{q} = \overline{\psi}\left(P_R \delta_{s,1} + P_L \delta_{s,N_5}\right)$$
$$P_{R/L} = \frac{1 \pm \gamma_5}{2}$$
$$(1 + \gamma_0 R) q(x) = 0, \quad \overline{q}(x)(1 - \gamma_0 R) = 0,$$

- **SF Dirichlet BC at fixed points**

$$(1 + \gamma_0) q(0) = 0, \quad (1 - \gamma_0) q(T) = 0$$
$$\overline{q}(0)(1 - \gamma_0) = 0, \quad \overline{q}(T)(1 + \gamma_0) = 0$$

- **Orbifolded action**

$$S = \frac{1}{2} \Sigma \overline{\psi} D_{\mathrm{SF}} \psi, \quad D_{\mathrm{SF}} = \frac{1 + A}{2} D_{\mathrm{dwf}} \frac{1 + A}{2}$$

Folding of temporal direction

- **Deleting unwanted degrees of freedom**

$$(1-A)\psi = 0 \quad \Leftrightarrow \quad \begin{cases} \psi(-x_0) = \gamma_0\gamma_5 PQ\psi(x_0) \\ (1 \mp \gamma_0\gamma_5 PQ)\psi(0/T) = 0 \end{cases}$$

- **Folded Dirac operator**

$$\begin{pmatrix} \Pi_+ \frac{B+C}{2}\Pi_+ & -\Pi_+ P_- & & & \\ -P_+\Pi_+ & B+C & -P_- & & \\ & -P_+ & B+C & -P_- & \\ & & -P_+ & B+C & -P_-\Pi_- \\ & & & -\Pi_- P_+ & \Pi_- \frac{B+C}{2}\Pi_- \end{pmatrix}_{x_0,y_0}$$

$$\Pi_\pm = \frac{1 \pm \gamma_0\gamma_5 PQ}{2}, \quad P_\pm = \frac{1 \pm \gamma_0}{2}, \quad [\Pi_\pm, P_\pm] \neq 0$$

$$B = (1-M) - B_W^{(3)} - \frac{1}{2}\left(\Omega^+ + \Omega^- - 2\right)$$

$$C = C_W^{(3)} + \gamma_5 \frac{1}{2}\left(\Omega^+ - \Omega^-\right)$$

D_{SF} : **invertible on** $\mathrm{diag}(\Pi_+, 1, 1, \Pi_-)$

Surface term

- **Inhomogeneous BC**

$$P_+ q(x)|_{x_0=0} = \rho(\vec{x}), \quad P_- q(x)|_{x_0=T} = \rho'(\vec{x})$$

$\Rightarrow$ **Act as source field**

$$\zeta(\vec{x}) = \frac{\delta}{\delta\overline{\rho}(\vec{x})}, \quad \overline{\zeta}(\vec{x}) = -\frac{\delta}{\delta\rho(\vec{x})}, \quad \mathcal{O} = \overline{\zeta}\gamma_5\zeta$$

$$Z_P(a/L) = C\frac{\sqrt{\langle\mathcal{O}\mathcal{O}'\rangle}}{\langle P(L/2)\mathcal{O}\rangle}$$

- **Local surface term**

$$\begin{aligned} S_{\text{surface}} \;=\; & -\overline{\rho}(\vec{x})P_- q(x)|_{x_0=0} - \overline{q}(x)P_+\rho(\vec{x})|_{x_0=0} \\ & -\overline{\rho}'(\vec{x})P_+ q(x)\big|_{x_0=N_T} - \overline{q}(x)P_-\rho'(\vec{x})\big|_{x_0=N_T} \end{aligned}$$

- **To reproduce continuum boundary propagator in $a \to 0$**

$$\langle q(x)\overline{\zeta}(\vec{y})\rangle = \langle q(x)\overline{q}(y)\rangle_{\text{SF}}\, P_+|_{y_0=0}$$

Formulation for dynamical simulation

- Ordinary DWF Dirac operator

$$D = \frac{D_{\rm dwf}}{D_{\rm PV}}$$

- Neuberger, Kikukawa-Noguchi $(N_5 \to \infty)$

$$\det \frac{D_{\rm dwf}}{D_{\rm PV}} = \det \frac{D_{\rm eff}}{D_{\rm eff}+1} = \det D_{\rm OD}, \qquad \frac{1}{D_{\rm eff}} = \langle q\bar{q} \rangle$$

- Strategy for SF DWF

  SF overlap Dirac operator : known

  $\Rightarrow$ find $D_{\rm eff}$ to reproduce $D_{\rm SF}^{\rm OD}$

- SF overlap Dirac operator

$$D_{\rm SF}^{\rm OD} = \left(\frac{1+\Gamma}{2} D_{\rm eff} \frac{1-\Gamma}{2} \right) \frac{1}{D_{\rm eff}+1}, \qquad \Gamma = \gamma_0 R$$

- **Caution**

$$D_{\rm SF}^{\rm OD} = \left(\frac{1+\Gamma}{2} D_{\rm eff} \frac{1-\Gamma}{2}\right) \frac{1}{D_{\rm eff}+1}$$

$D_{\rm SF}^{\rm OD}$ is not Hermite

Complexity is $\mathcal{O}(a)$ effect for overlap Dirac op

- **Two flavors system $\Rightarrow$ for second flavor**

$$\widetilde{D}_{\rm SF}^{\rm OD} = \frac{1}{D_{\rm eff}+1}\left(\frac{1+\Gamma}{2} D_{\rm eff} \frac{1-\Gamma}{2}\right)$$

$$\det D_{\rm SF}^{\rm OD} \widetilde{D}_{\rm SF}^{\rm OD} = \det D_{\rm SF}^{\rm OD} \left(D_{\rm SF}^{\rm OD}\right)^{\dagger}$$

$$\left(\frac{1}{D_{\rm SF}^{\rm OD}}\right)^{\dagger} \frac{1}{D_{\rm SF}^{\rm OD}} = \frac{1+\Gamma}{2}\left(\frac{1}{D_{\rm eff}^{\dagger}}+1\right)\left(\frac{1}{D_{\rm eff}}+1\right)\frac{1+\Gamma}{2}$$

for massless case