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# Reggeized Gluon States in Multi-colour QCD: Pomeron vs Odderon

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## Compound Reggeon states in QCD:

$N = 2$: BFKL Pomeron (1978)
$N = 3$: QCD odderon (Janik & Wosiek 1997)
$N \geq 4$: ...

The first calculation of the intercept of multi-Reggeon states in multi-colour QCD

G. P. Korchemsky, J. K., A. N. Manashov, Phys. Rev. Lett. **88** 122002
S. É. Derkachov, G. P. Korchemsky, J. K., A. N. Manashov, *Nucl. Phys.* **B645**, 237 (2002)
G. P. Korchemsky, J. K., A. N. Manashov, *Phys. Lett.* **B583**, 121 (2004)
J. K., *Acta. Phys. Pol.* **B33**, 3621 (2002)
J. K., M. Praszałowicz, Acta Phys. Pol. **B33** 665 (2002)
J. K.– PhD thesis

# Multi-Reggeon Equation (BKP)

The scattering amplitude in the Regge limit: $s \to \infty, \quad t = const$

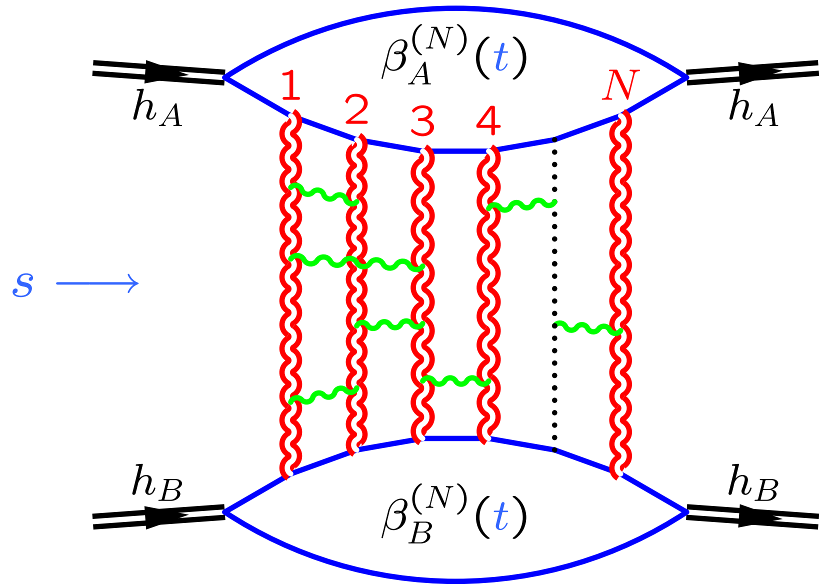
$$\boxed{A(s,t) \sim -i \sum_{N=2}^{\infty} (i\bar{\alpha}_s)^N \frac{s^{1-\bar{\alpha}_s E_N}}{\sqrt{\bar{\alpha}_s \ln s}} \beta_A^{(N)}(t)\beta_B^{(N)}(t)}$$

$\bar{\alpha}_s = \alpha_s N_c/\pi$

The compound $N$-Reggeon states satisfy the Schrödinger equation:

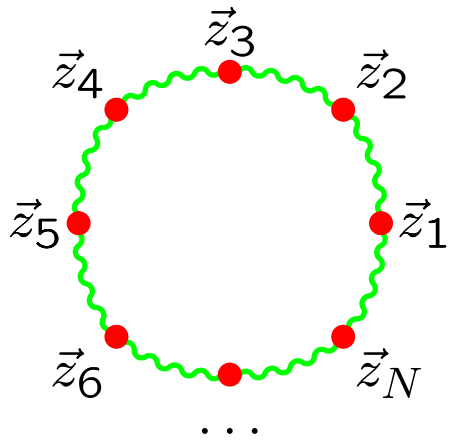
$$\boxed{\mathcal{H}_N \Psi(\{\vec{z}_k\}) = E_N \Psi(\{\vec{z}_k\})}$$

$E_N$ - the "energy" of the ground states



# QCD Hamiltonian in the Multi-Colour Limit: $N_c \to \infty$

$$\mathcal{H}_N = \sum_{k=1}^{N} H(\vec{z}_k, \vec{z}_{k+1})$$




where $\vec{z}_1 \equiv \vec{z}_{N+1}$

- has a hidden symmetry,

- possesses the set of the integrals of motion (integrable):

$$[\mathcal{H}_N, \widehat{q}_n] = [\widehat{q}_n, \widehat{q}_m] = 0, \qquad q_2 = -h(h-1) \qquad n, m = 2, \ldots, N$$

- is equivalent to the XXX Heisenberg spin magnet

- separates in (anti)holomorphic parts

# Pomeron vs Odderon

Pure gluon: $\boxed{A_\mu(x) \xrightarrow{C} -A_\mu^T(x)}$ where $A_\mu(x) = A_\mu^a(x) t^a$

$N = 2$:

$\mathbb{P}_{\mu\nu}(x, y) = \mathrm{tr}(A_\mu(x) A_\nu(y)) = \frac{1}{2}\delta_{ab} A_\mu^a(x) A_\nu^b(y)$ $\boxed{C = +1}$

$N = 3$:

$\mathbb{P}_{\mu\nu\rho}(x, y, z) = -i\,\mathrm{tr}([A_\mu(x), A_\nu(y)] A_\rho(z)) = \frac{1}{2} f_{abc} A_\mu^a(x) A_\nu^b(y) A_\rho^c(z)$ $\boxed{C = +1}$

$\mathbb{O}_{\mu\nu\rho}(x, y, z) = \mathrm{tr}(\{A_\mu(x), A_\nu(y)\} A_\rho(z)) = \frac{1}{2} d_{abc} A_\mu^a(x) A_\nu^b(y) A_\rho^c(z)$ $\boxed{C = -1}$

Reggeons:

$$\begin{cases} \mathbb{P}: & f_{abc} \Leftrightarrow \Psi^{(-)}(\vec{z}_1, \vec{z}_2, \vec{z}_3) \\ \mathbb{O}: & d_{abc} \Leftrightarrow \Psi^{(+)}(\vec{z}_1, \vec{z}_2, \vec{z}_3) \end{cases}$$ where under mirror permutation

$$\mathbb{M}\Psi(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N) = \Psi(\vec{z}_N, \ldots, \vec{z}_2, \vec{z}_1)$$

$$\mathbb{M}\Psi^{(\pm)}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N) = \pm\Psi^{(\pm)}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)$$

$$\Psi^{(\pm)}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N) = \frac{1}{2}\left[\Psi_{\{q,\bar{q}\}}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N) \pm \Psi_{\{-q,-\bar{q}\}}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)\right]$$

where $q = \{q_k\}, -q = (-1)^k q_k$

# Wave-Functions for Three Reggeons

$$\Psi \equiv \Psi_{\{q_3,\bar{q}_3\}}(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0)$$

Lipatov's ansatz:

$$\boxed{\hat{q}_2 \Psi = -(h-1)h\Psi \quad \& \quad (\textit{a-h. sec.})} \quad \Longrightarrow \quad \boxed{\Psi = w^h \bar{w}^{\bar{h}} F(x,\bar{x})}$$

where $w = \frac{(z_3 - z_2)}{(z_3 - z_0)(z_2 - z_0)}$ and $x = \frac{(z_1 - z_2)(z_3 - z_0)}{(z_1 - z_0)(z_3 - z_2)}$

Janik-Wosiek solution:

$$\boxed{\hat{q}_3 \Psi = q_3 \Psi \quad \& \quad (\textit{a-h. sec.})} \quad \Longrightarrow \begin{cases} \text{spectrum of } \hat{q}_3 \\ \Psi_{\{q_3,\bar{q}_3\}}(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0) \end{cases}$$

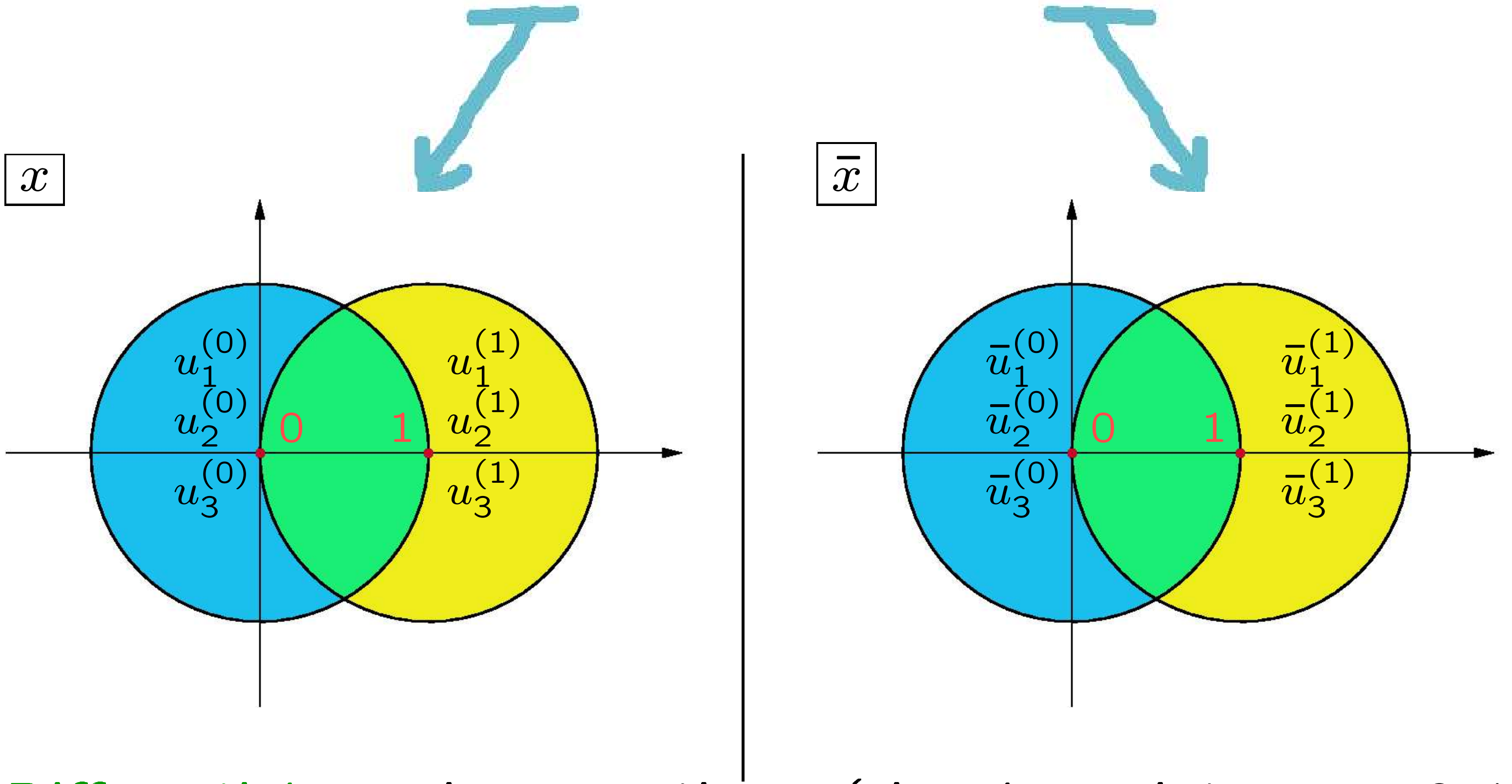
where $\bar{q}_3 = q_3^*$ and $\bar{h} = 1 - h^*$ with $h = \frac{1+n_h}{2} + i\nu_h$ and $n_h \in \mathbb{Z}$, $\nu_h \in \mathbb{R}$

Usually, the wave-functions $\Psi_{\{q_3,\bar{q}_3\}}(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0)$ have mixed $C-$parity, so they contain Pomeron as well as odderon contribution.

# Janik-Wosiek solution

$$\Psi_{\{q_3,\bar{q}_3\}}(\vec{z}_1,\vec{z}_2,\vec{z}_3;\vec{z}_0) = w^h \bar{w}^{\bar{h}} F(x,\bar{x})$$

$$\boxed{F(x,\bar{x}) = \sum_{n,\bar{n}=1}^{3} u_n^{(i)}(x;q_3) A_{n,\bar{n}}^{(i)}(q_3,\bar{q}_3) \bar{u}_{\bar{n}}^{(i)}(\bar{x};\bar{q}_3)}$$

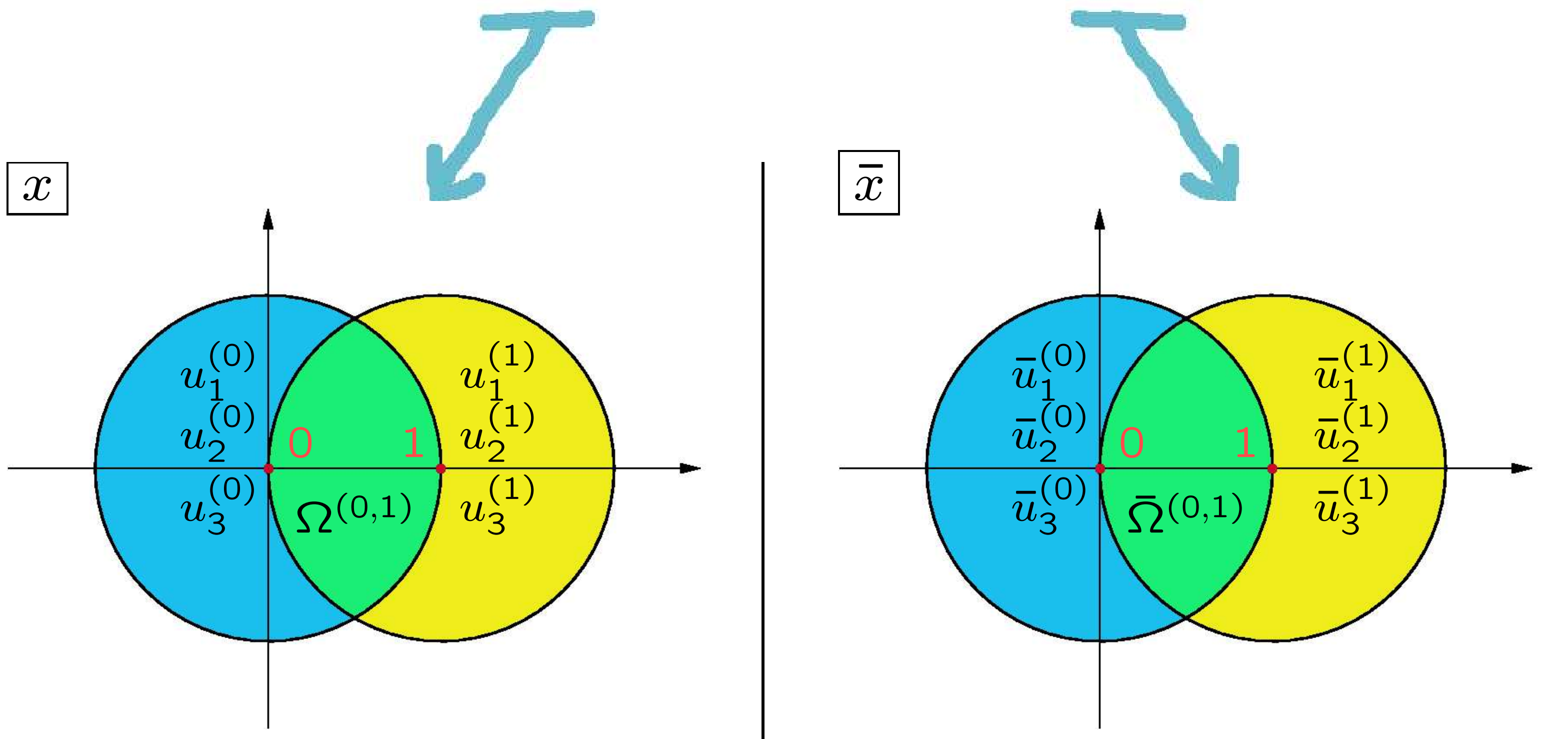



Differential $q_3-$eigenequation (singular points: $x_i = 0, 1, \infty$)

# Janik-Wosiek solution

$$\Psi_{\{q_3,\bar{q}_3\}}(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0) = w^h \bar{w}^{\bar{h}} F(x, \bar{x})$$

$$\boxed{F(x,\bar{x}) = \sum_{n,\bar{n}=1}^{3} u_n^{(i)}(x; q_3) A_{n,\bar{n}}^{(i)}(q_3, \bar{q}_3) \bar{u}_{\bar{n}}^{(i)}(\bar{x}; \bar{q}_3)}$$




Transition matrices $\Omega$: $u_m^{(i)}(x; q_3) = \sum_{n=1}^{3} \Omega_{mn}^{(i,j)}(q_3) u_n^{(j)}(x; q_3)$

# Janik-Wosiek solution

$$\Psi_{\{q_3,\bar{q}_3\}}(\vec{z}_1,\vec{z}_2,\vec{z}_3;\vec{z}_0) = w^h \bar{w}^{\bar{h}} F(x,\bar{x})$$

$$\boxed{F(x,\bar{x}) = \sum_{n,\bar{n}=1}^{3} u_n^{(i)}(x;q_3) A_{n,\bar{n}}^{(i)}(q_3,\bar{q}_3) \bar{u}_{\bar{n}}^{(i)}(\bar{x};\bar{q}_3)}$$

Differential $q_3-$eigenequation

(singular points: $x_i = 0, 1, \infty$)

Three series solutions $u_n^{(i)}(x;q_3)$ for each $x_i$ with $\mathrm{Log}(x)$ and $x^a$

Transition matrices $\Omega$: $u_m^{(i)}(x;q_3) = \sum_{n=1}^{3} \Omega_{mn}^{(i,j)}(q_3) u_n^{(j)}(x;q_3)$

and similarly for antiholomorphic sector

$$\boxed{A^{(j)}(q_3,\bar{q}_3) = \Omega^{(i,j)}(q_3)^T \cdot A^{(i)}(q_3,\bar{q}_3) \cdot \bar{\Omega}^{(i,j)}(\bar{q}_3)}$$

and singlevaluedness of $F(x,\bar{x})$

give quantization conditions for $q_3$ and structure of $A^{(j)}(q_3,\bar{q}_3)$

# Wave-Functions for Three Reggeons

$$\Psi \equiv \Psi_{\{q_3=0,\bar{q}_3=0\}}(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0)$$

It is possible to resum series solutions for $q_3 = 0$:

Pomeron $q_3 = 0$ with $E_3 > 0$:

$(h, \bar{h}) = (1, 0)$ or $(0, 1)$

$$\boxed{\Psi(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0) = w^h \bar{w}^{\bar{h}} \left(1 + (-x)^h (-\bar{x})^{\bar{h}} + (x-1)^h (\bar{x}-1)^{\bar{h}}\right)} = 0$$

Pomeron $q_3 = 0$ i $(h, \bar{h}) = (1, 0)$ with $E_3 = 0$:

$$\boxed{\Psi(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0) = w\left((-x)\,\mathrm{Log}(x\bar{x}) + (x-1)\,\mathrm{Log}((x-1)(\bar{x}-1))\right)}$$

odderon $q_3 = 0$ (BLV solution) with $E_3 \geq 0$:

$$\boxed{\Psi(\vec{z}_1, \vec{z}_2, \vec{z}_3; \vec{z}_0) = w^h \bar{w}^{\bar{h}} x(x-1)\bar{x}(\bar{x}-1)\left(\delta^{(2)}(x) + \delta^{(2)}(1-x) + x^{h-3}\bar{x}^{\bar{h}-3}\delta^{(2)}\left(x^{-1}\right)\right)}$$

# The Baxter $Q$-Operator: $\mathbb{Q}(u,\bar{u})$

The $Q$-operator has to satisfy the conditions:

Commutation

$$[\mathbb{Q}(u,\bar{u}),\mathbb{Q}(v,\bar{v})] = [\hat{t}_N(u),\mathbb{Q}(v,\bar{v})] = [\hat{t}_N(\bar{u}),\mathbb{Q}(v,\bar{v})] = 0,$$
$$\hat{t}_N(u) = 2u^N + \hat{q}_2 u^{N-1} + \ldots + \hat{q}_N$$

Baxter equations

$$\hat{t}_N(u)\mathbb{Q}(u,\bar{u}) = u^N\mathbb{Q}(u+i,\bar{u}) + u^N\mathbb{Q}(u-i,\bar{u})$$
$$\hat{t}_N(\bar{u})\mathbb{Q}(u,\bar{u}) = (\bar{u}+i)^N\mathbb{Q}(u,\bar{u}+i) + (\bar{u}-i)^N\mathbb{Q}(u,\bar{u}-i)$$

$Q-$operator has prescribed analytical properties (known pole structure) and asymptotic behaviour at infinity

$$\boxed{\mathcal{H}_N\left[\mathbb{Q}_{\hat{q}_k}(u,\bar{u})\right]} \longrightarrow \boxed{\text{Energy Spectrum}}$$

# Our Method

Baxter equations

QCD $\mathcal{H}_N$

$N_c \to \infty$

the integral ansatz for

$\mathbb{Q}(u,\overline{u}) = \int Q_{\vec{q}}(z,\overline{z}) K(u,\overline{u},z,\overline{z}) d^2 z$

$\mathcal{H}_N = \sum_i H(\vec{z}_k, \vec{z}_{k+1})$

Differential equations for $Q_{\vec{q}}(z,\overline{z})$

the integrals of motion $\widehat{\vec{q}}_k$

singlevaluedness of $Q_{\vec{q}}(z,\overline{z})$ in the complex $z$-plain

$\mathcal{H}_N(\widehat{\vec{q}})$

Quantum conditions for the eigenvalues of $\widehat{\vec{q}}_k$

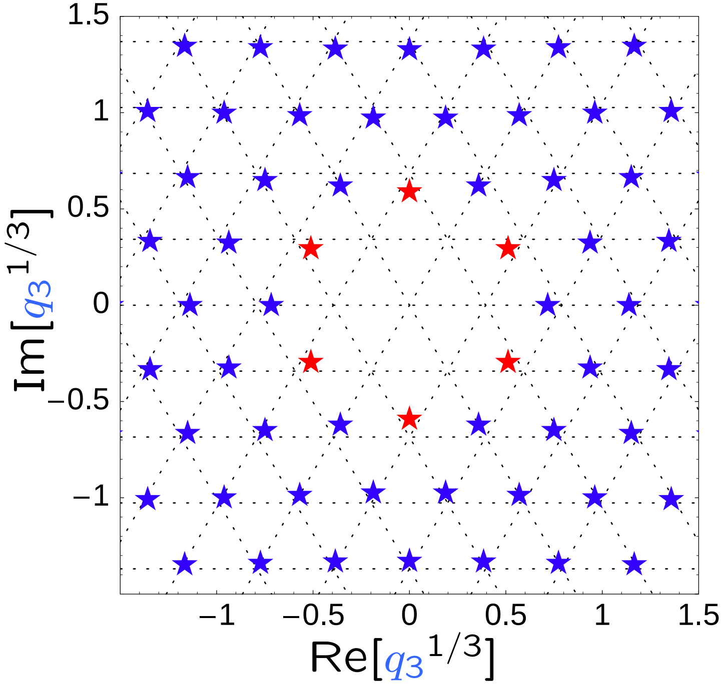
the Baxter $\mathbb{Q}$-operator

$\mathcal{H}_N\left[\widehat{\mathbb{Q}}_{\vec{q}}(u,\overline{u})\right]$

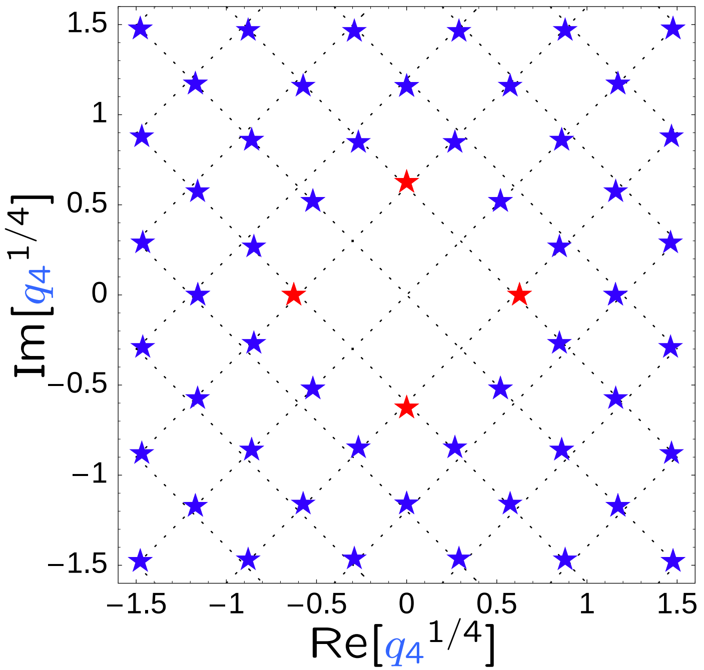
Spectrum of the energy

# The Results: the eigenvalues of $\hat{q}_k$ for $N = 3, 4$

$N = 3$: $h = \frac{1}{2}$




$N = 4$: $h = \frac{1}{2}$, $q_3 = 0$




$$[q_3^{\text{approx}}(\ell_1, \ell_2)]^{1/3} = \frac{\Gamma^3(2/3)}{2\pi}\left(\frac{1}{2}\ell_1 + i\frac{\sqrt{3}}{2}\ell_2\right)$$

$$[q_4^{\text{approx}}(\ell_1, \ell_2)]^{1/4} = \frac{\Gamma^2(3/4)}{2\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\ell_1 + i\frac{1}{\sqrt{2}}\ell_2\right)$$

where $\ell_1, \ell_2 \in \mathbb{Z}$ and $\ell_1 + \ell_2$ is even

$$\boxed{h = \frac{1+n_h}{2} + i\nu_h}$$

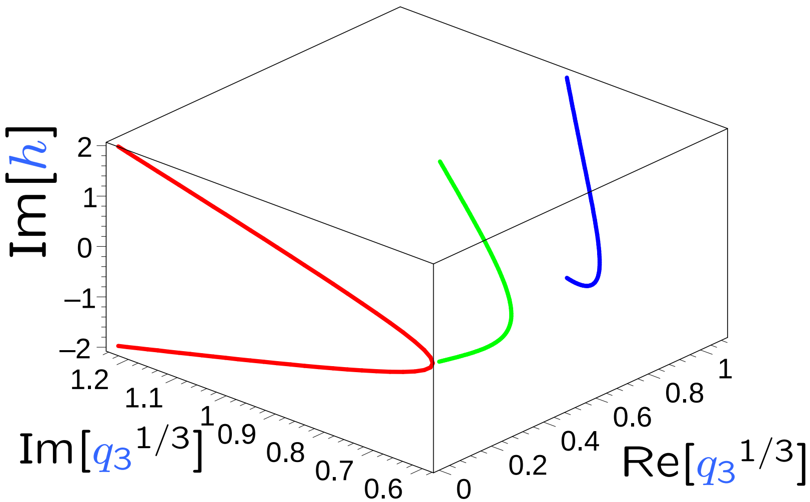
$$\boxed{q_2 = -h(h-1)}$$

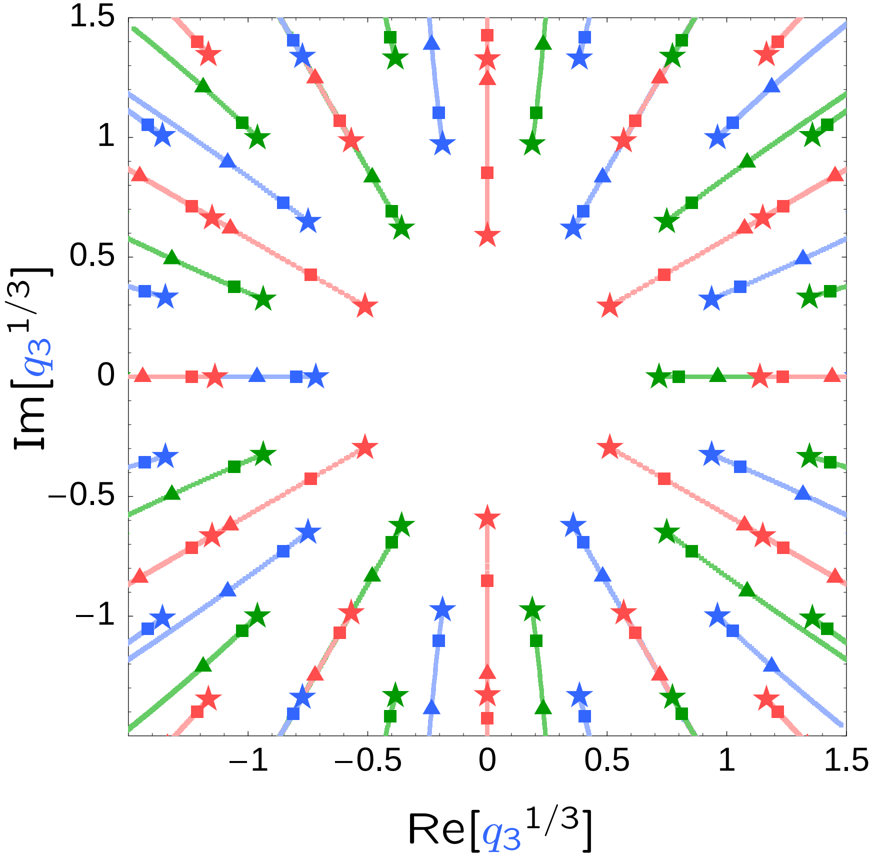
Similarly for higher $N$'s

$n_h \in \mathbb{Z}$, $\nu_h \in \mathbb{R}$

# The Results: continuation in $\nu_h$ and quasimomenta for $N = 3$

$q_2 = \frac{1}{4} + i\,{\nu_h}^2$, $\nu_h \in \mathbb{R}$




Possible quasimomenta

$\theta_3 = 0$ RED

$\theta_3 = \frac{2\pi}{3}$ BLUE

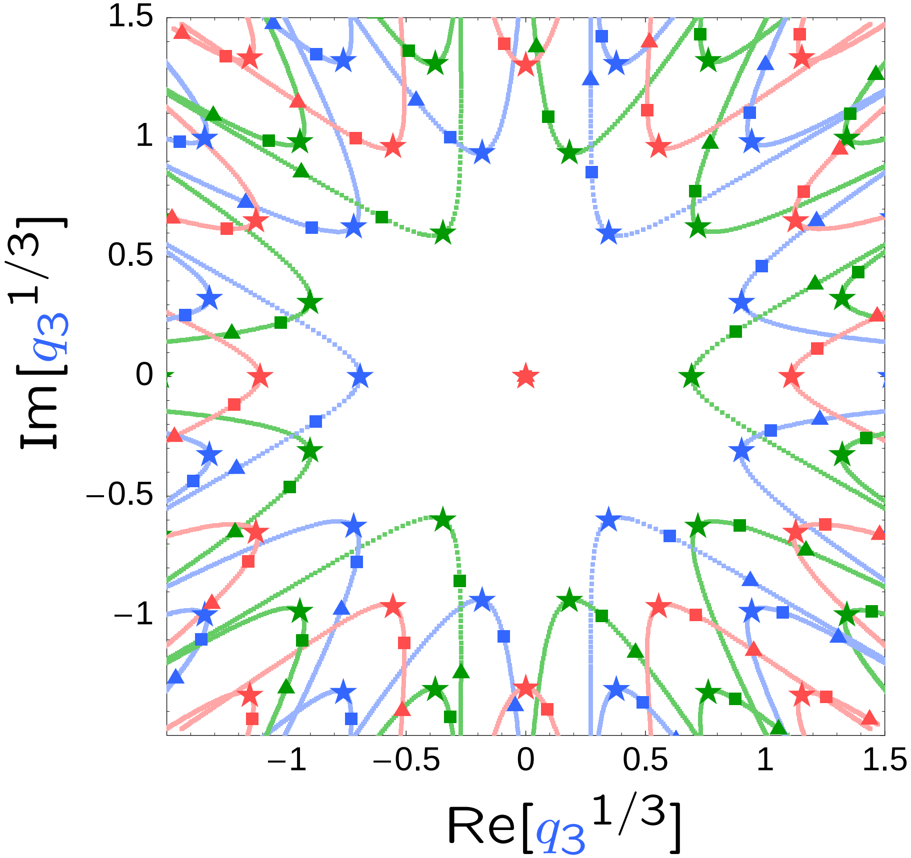
$\theta_3 = \frac{4\pi}{3}$ GREEN

$$\boxed{\mathbb{P}\Psi_{\vec{q}}(\{\vec{z}_k\}) = e^{i\theta_N(\vec{q})}\Psi_{\vec{q}}(\{\vec{z}_k\})}$$

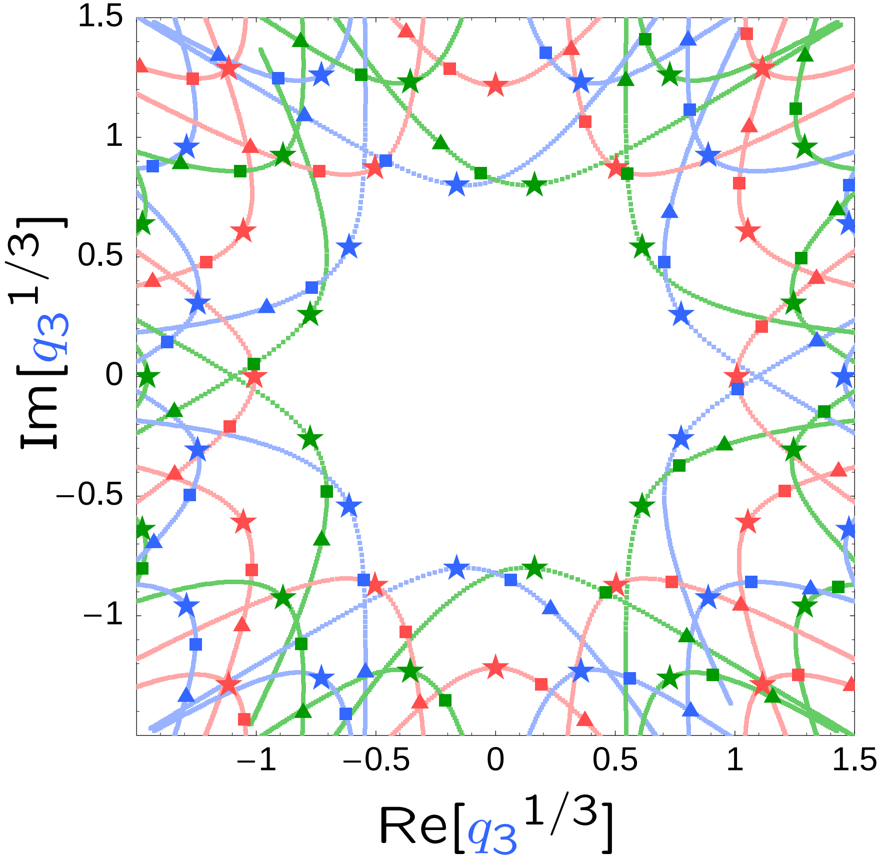
where $\mathbb{P}\Psi\left(\vec{z}_1, \ldots, \vec{z}_{N-1}, \vec{z}_N\right) = \Psi\left(\vec{z}_2, \ldots, \vec{z}_N, \vec{z}_1\right)$

# Continuation in $\nu_h$ and spectrum rotation for $|n_h| > 0$ and $N = 3$

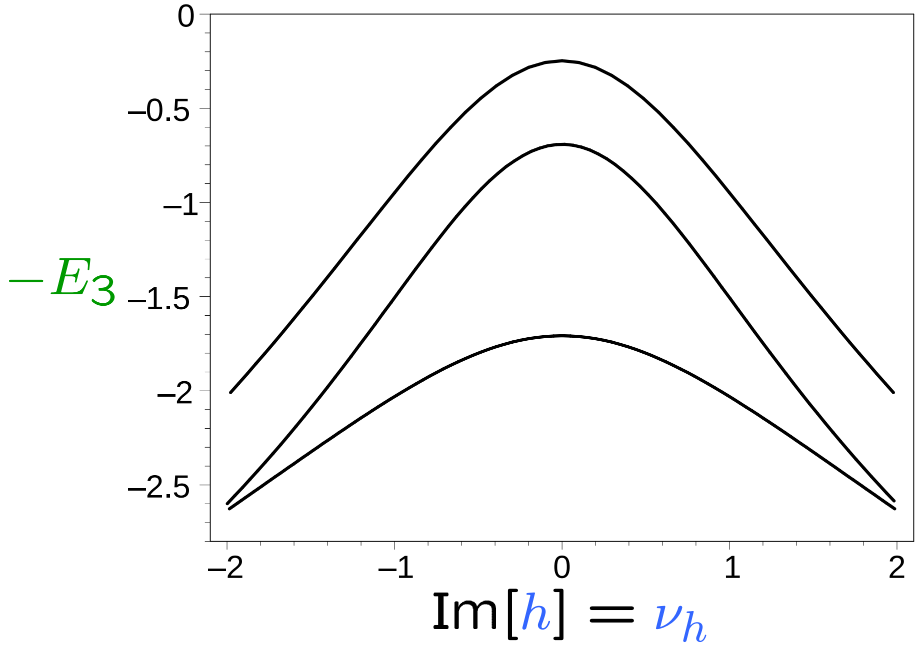
$h = 1 + i\nu_h \ (n_h = 1)$




$h = \frac{3}{2} + i\nu_h \ (n_h = 2)$



# Energy along trajectories for $N = 3$ and $h = 1/2$



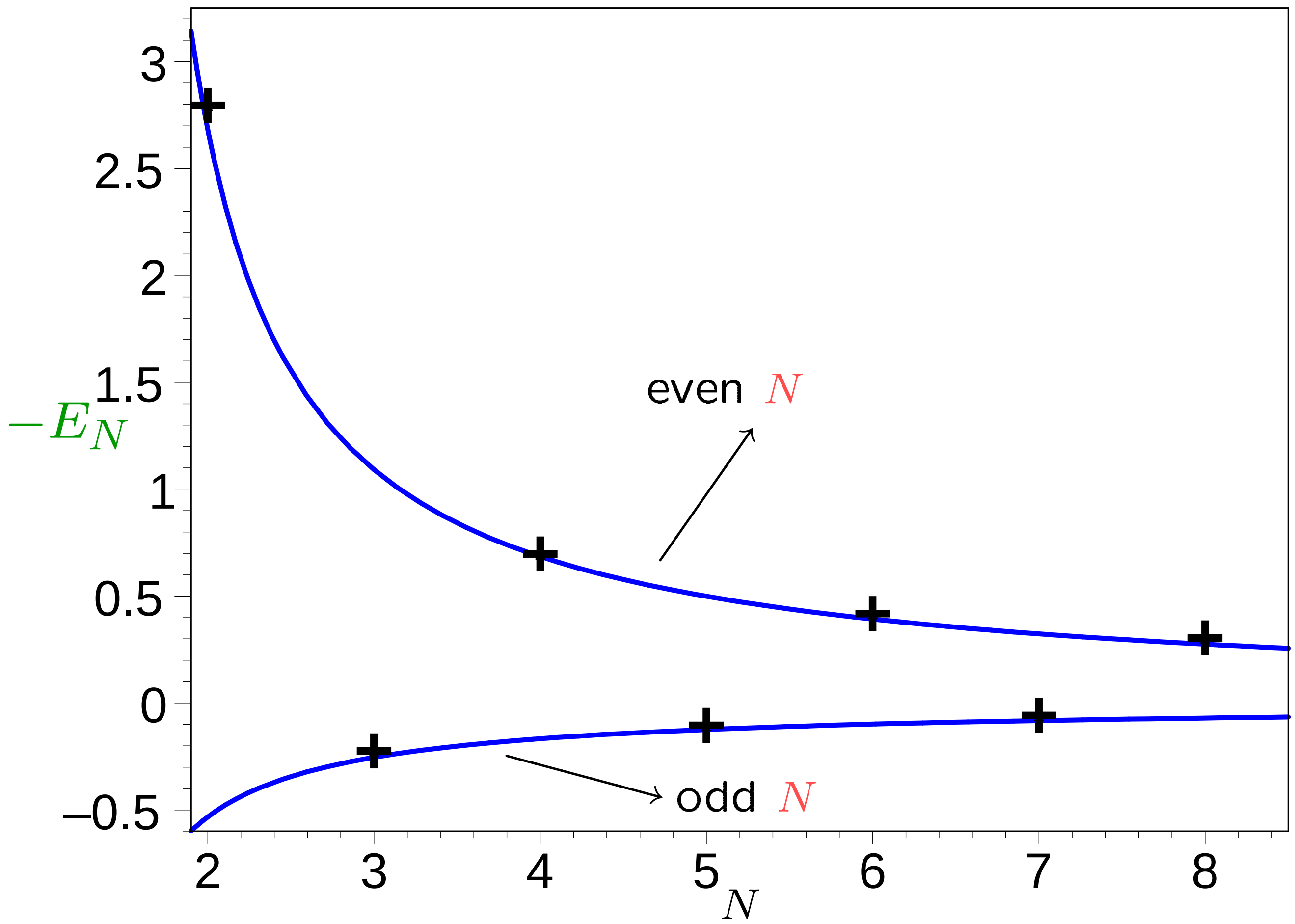

| $(\ell_1, \ell_2)$ | $\left(q_3{}^{\text{exact}}\right)^{1/3}$ | $\left(q_3{}^{\text{WKB}}\right)^{1/3}$ | $-E_3$ |
|---|---|---|---|
| $(0, 2)$ | $0.590\, i$ | $0.684\, i$ | $-0.2472$ |
| $(2, 2)$ | $0.358 + 0.621\, i$ | $0.395 + 0.684\, i$ | $-0.6910$ |
| $(4, 2)$ | $0.749 + 0.649\, i$ | $0.790 + 0.684\, i$ | $-1.7080$ |
| $(6, 2)$ | $1.150 + 0.664\, i$ | $1.186 + 0.684\, i$ | $-2.5847$ |
| $(8, 2)$ | $1.551 + 0.672\, i$ | $1.581 + 0.684\, i$ | $-3.3073$ |
| $(10, 2)$ | $1.951 + 0.676\, i$ | $1.976 + 0.684\, i$ | $-3.9071$ |

# Compound $N$ Reggeon States in Multi-Colour QCD

| $N$ | $iq_3$ | $q_4$ | $iq_5$ | $q_6$ | $iq_7$ | $q_8$ | $-E_N$ |
|---|---|---|---|---|---|---|---|
| 2 | | | | | | | 2.7726 |
| 3 | 0.20526 | | | | | | -0.2472 |
| 4 | 0 | 0.15359 | | | | | 0.6742 |
| 5 | 0.26768 | 0.03945 | 0.06024 | | | | -0.1275 |
| 6 | 0 | 0.28182 | 0 | 0.07049 | | | 0.3946 |
| 7 | 0.31307 | 0.07099 | 0.12846 | 0.00849 | 0.01950 | | -0.0814 |
| 8 | 0 | 0.39117 | 0 | 0.17908 | 0 | 0.03043 | 0.2810 |

$$E_N = E_N(q_3, q_4, \ldots, q_N) \qquad h = \frac{1}{2}$$

# The Dependence of $E_N$ on $N$ for $h = \frac{1}{2}$




The upper curve:

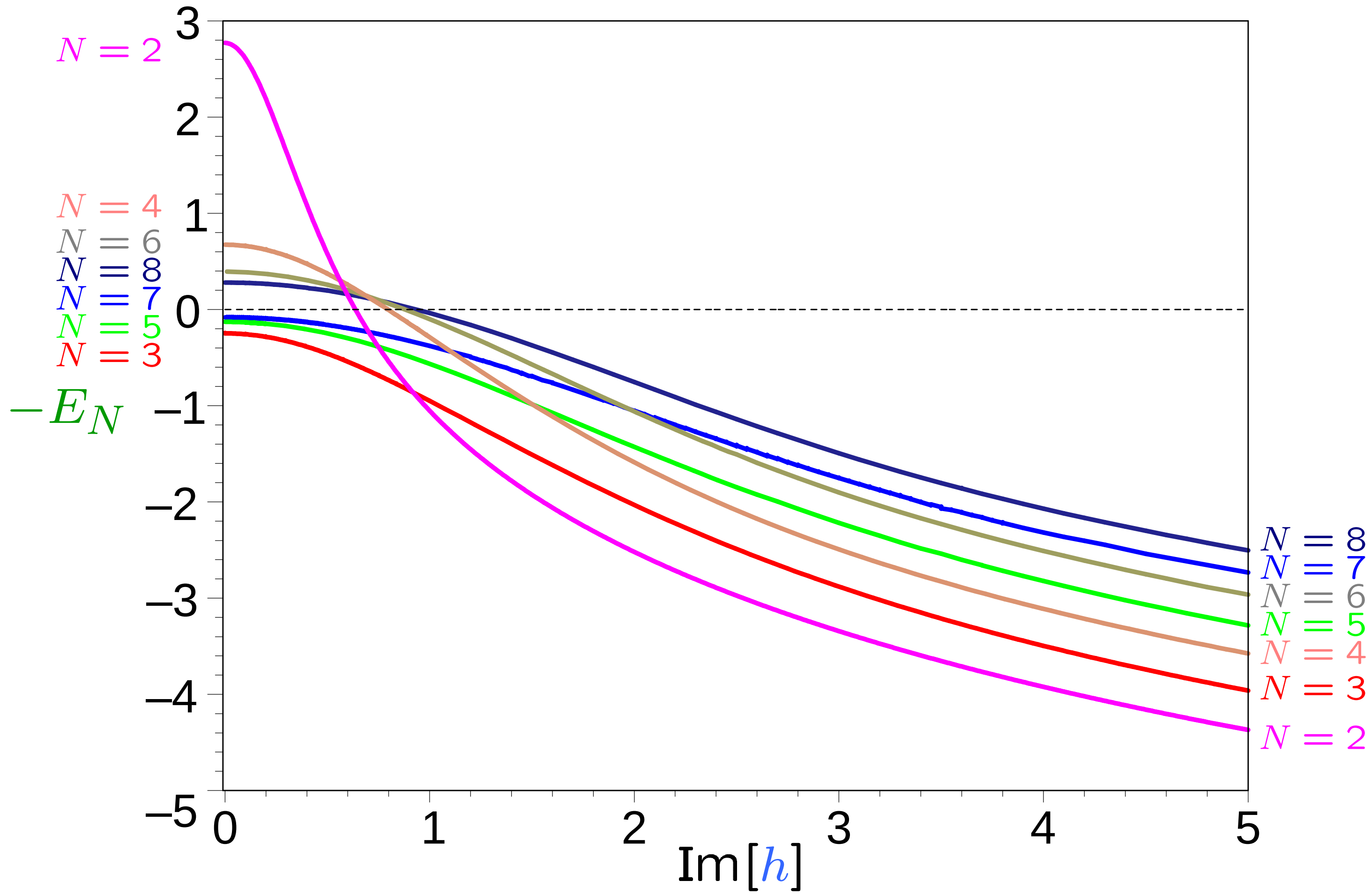
$1.8402/(N - 1.3143)$

The lower curve:

$-2.0594/(N - 1.0877)$

Energy is negative for odd $N$ and positive for even $N$

# The Ground Branches $E_N(\mathrm{Im}(h))$ for $h = \frac{1}{2} + i\nu_h$



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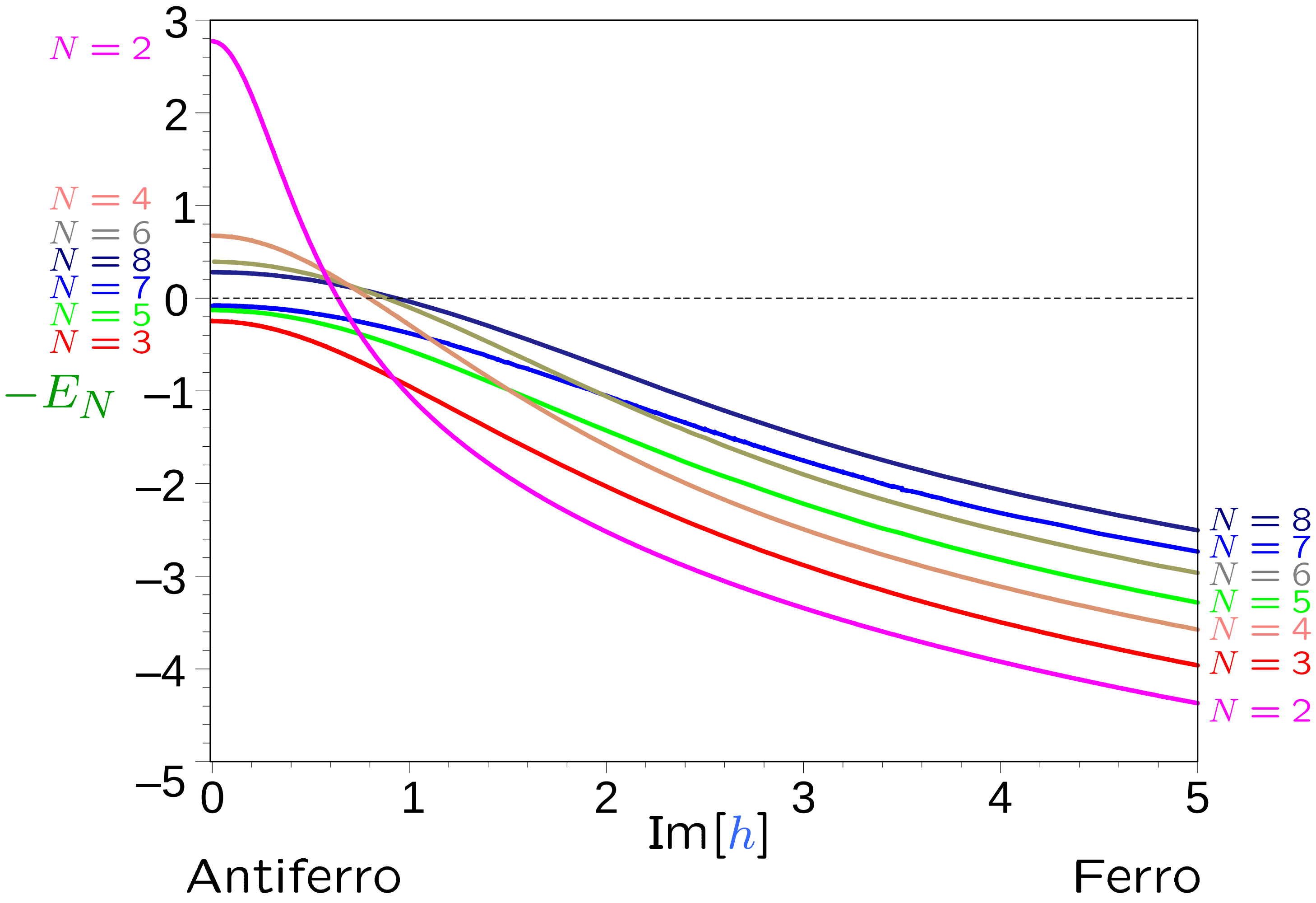

# The Summary

- We found the spectrum of multi-Reggeon compound states in QCD
- The $q_3-$eigenequation for $N = 3$ Reggeons
  - wave-functions, conformal charges, intercepts $(= 1 - \bar{\alpha}_s \text{min} E_N)$
  - $q_3 \neq 0 \Rightarrow$ mixed $C-$parity
  - $q_3 = 0$: Pomeron and odderon with intercept $\alpha_3 = 1$
- Higher $N$: $Q-$Baxter method
  - equivalent method to the $q_3-$eigenequation method for $N = 3$
  - lattice-like $q_N-$spectrum
  - formulae for the Reggeon wave-functions more complicated
  - $q_N = 0$ descendent states for odd N with $E_N = 0$
- Possible Future Work:
  - Calculations of $A(s, t)$
  - Analytical Explanations

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Properties of the states
– winding lattices
– resemblant lattices
– descendent states
– anomalous dimensions

# Lipatov's quantization conditions

$\mathcal{H}\Psi(u,\bar{u}) = E\Psi(u,\bar{u})$ with $\Psi(u,\bar{u}) = c_1\Psi_1(u,\bar{u}) + c_2\Psi_2(u,\bar{u})$

where $\begin{array}{l} \mathbb{Q}\Psi_1(u,\bar{u}) = \mathcal{Q}^{(1)}\Psi_1(u,\bar{u}) \\ \mathbb{Q}\Psi_2(u,\bar{u}) = \mathcal{Q}^{(2)}\Psi_2(u,\bar{u}) \end{array}$ and $\begin{array}{l} \Psi_1(u,\bar{u}), \Psi_2(u,\bar{u}) - \text{independent} \\ [\mathbb{Q}, \mathcal{H}] = 0 \end{array}$

then $\begin{array}{l} \mathcal{H}\Psi_1(u,\bar{u}) = E^{(1)}\Psi_1(u,\bar{u}) \\ \mathcal{H}\Psi_2(u,\bar{u}) = E^{(2)}\Psi_2(u,\bar{u}) \end{array}$ and $E^{(1)} = E^{(2)}$

However, $E^{(1)}(q_2, q_3, \ldots, q_N) = E^{(2)}(q_2, q_3, \ldots, q_N)$ for all $\{q_2, q_3, \ldots, q_N\}$
This does not produce any additional quantization conditions for $q_k$

Problem: $E^{(1)} = \sum_{n=0}^{\infty} a_n$ and $E^{(2)} = \sum_{n=0}^{\infty} b_n$

where $\sum_{n=0}^{M} a_n \neq \sum_{n=0}^{M} b_n$ for $M < \infty$

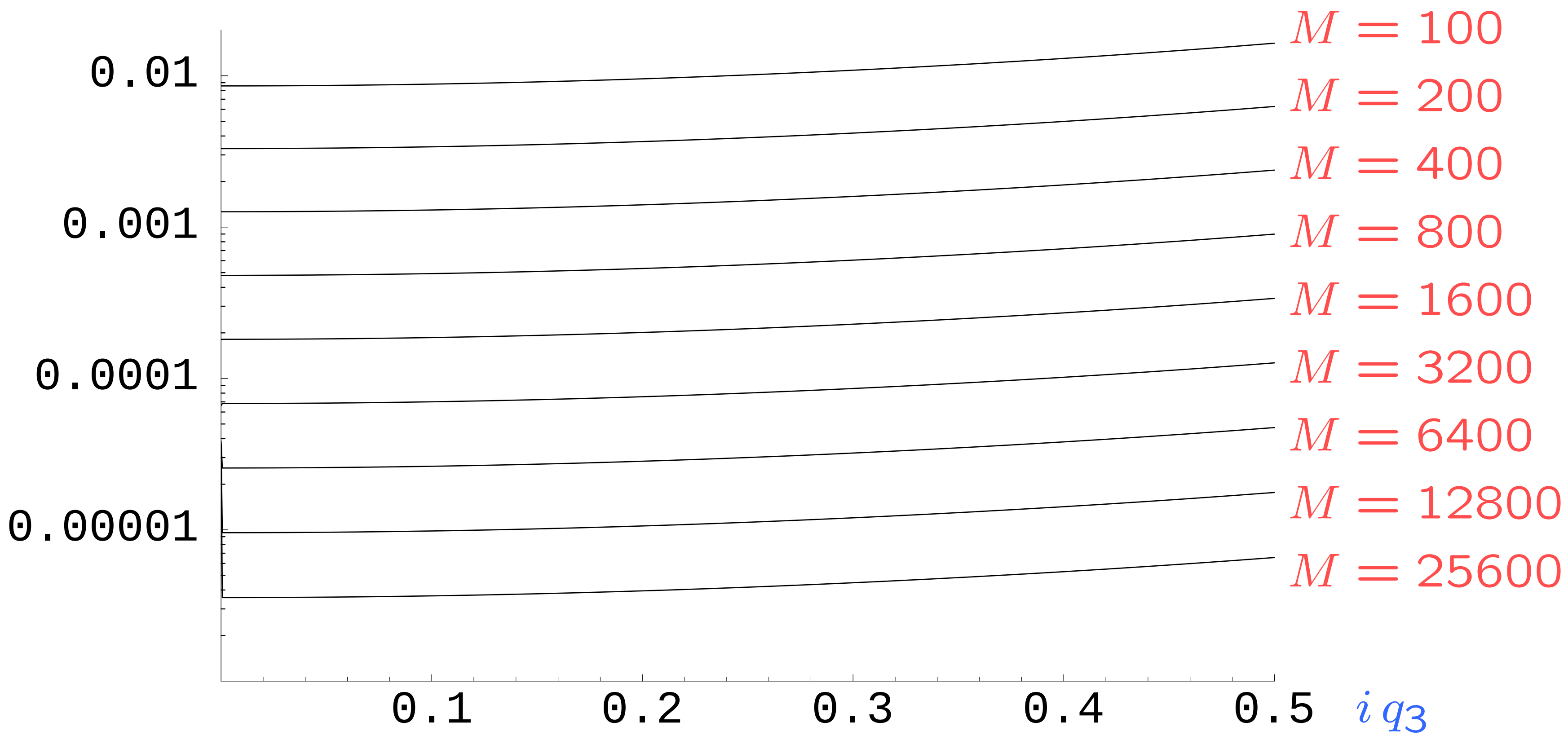
$E^{(i)} = \varepsilon^{(i)} + \bar{\varepsilon}^{(i)}$ and $\Psi_i(u,\bar{u}) = \Psi_i(u)\Psi_i(\bar{u})$ with $\mathcal{H}\Psi_i(u) = \varepsilon^{(i)}\Psi_i(u)$

$\boxed{\text{Lipatov's q.cond.: } \varepsilon^{(1)} = \varepsilon^{(2)}}$ (but $\not\Leftarrow$) $E^{(1)} = E^{(2)}$ .

# Lipatov's $E_3^{(1)} - E_3^{(2)}$ for $h = 1/2$

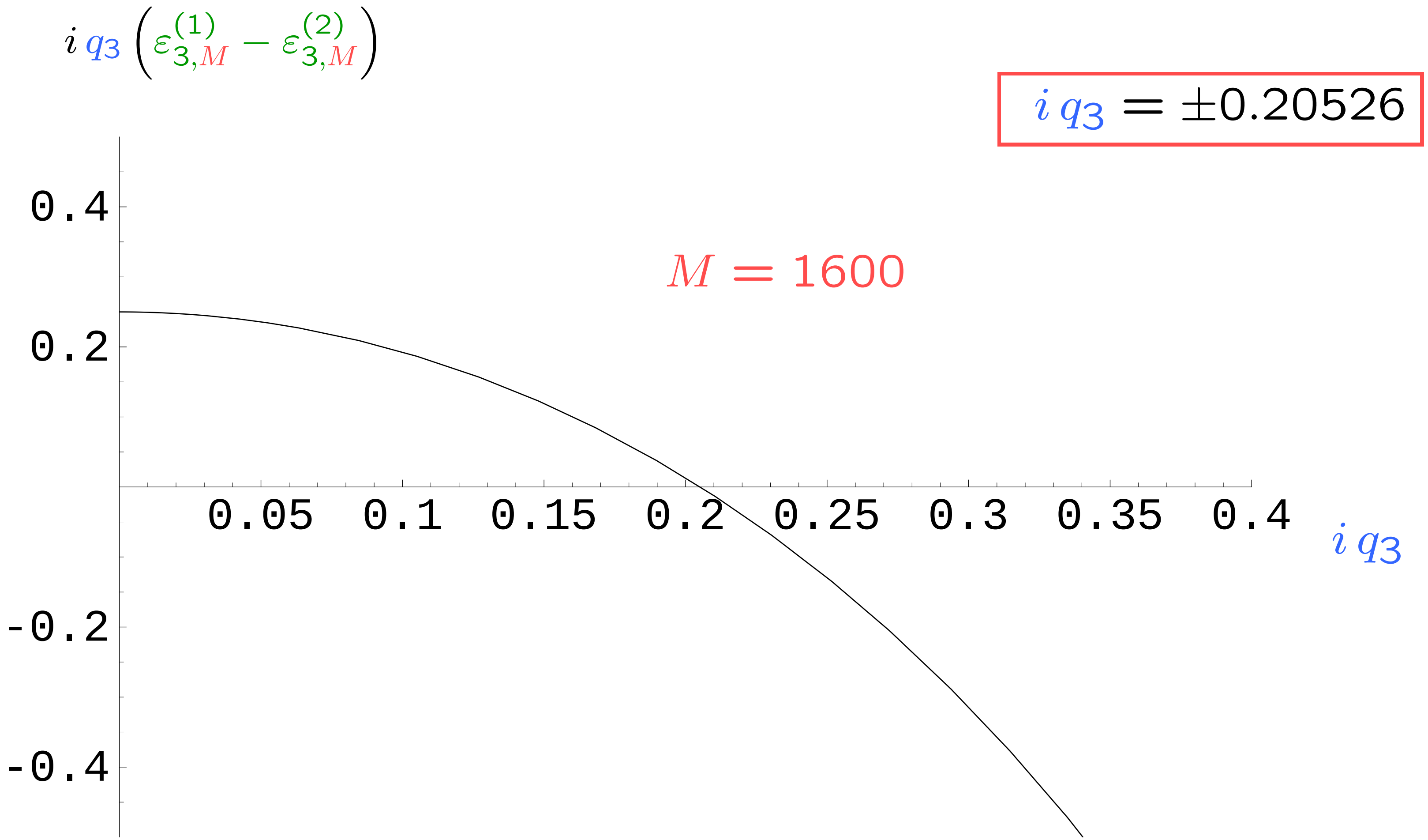
$$\sum_{n=0}^{M} a_n - \sum_{n=0}^{M} b_n$$

$\boxed{i\, q_3 = \pm 0.20526}$




where $E_3^{(1)} = \sum_{n=0}^{\infty} a_n$ and $E_3^{(2)} = \sum_{n=0}^{\infty} b_n$ while $E_3^{(1)} = E_3^{(2)}$

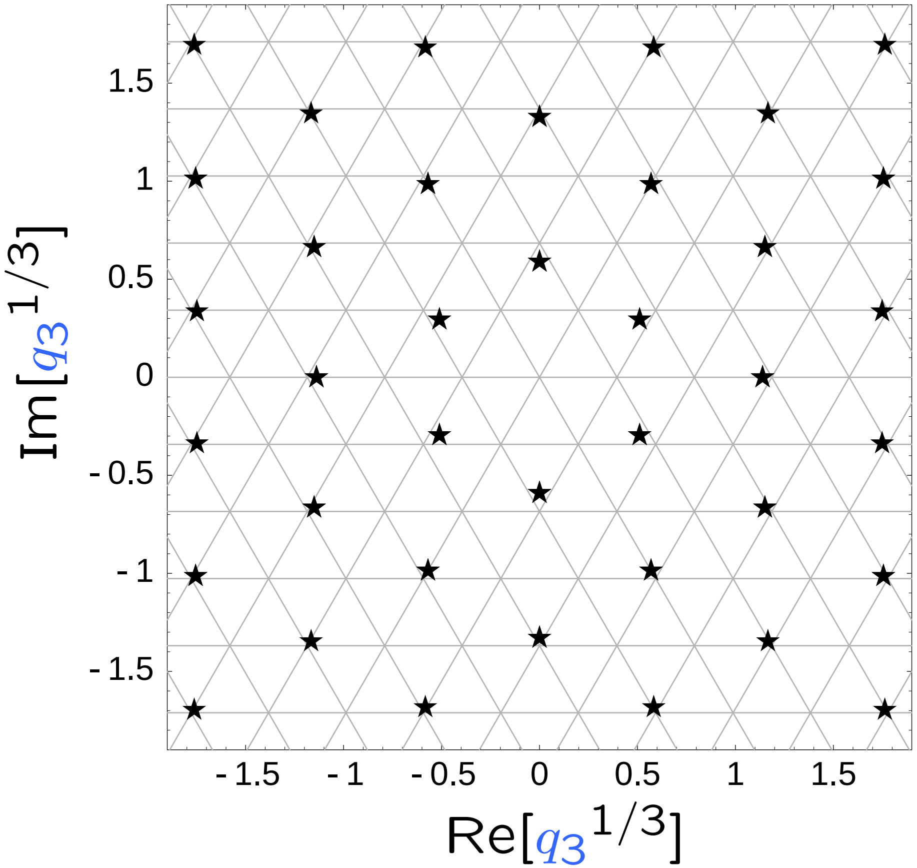
Real Lipatov's test function: $\varepsilon_3^{(1)} - \varepsilon_3^{(2)}$ for $h = 1/2$



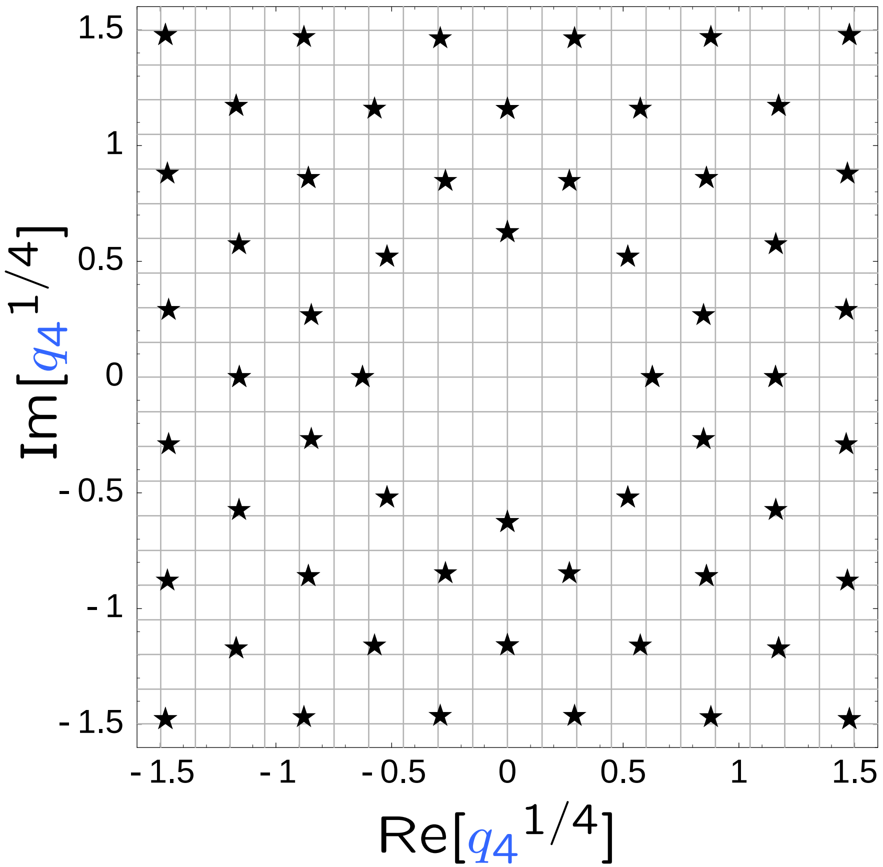

where $E_3^{(i)} = \lim_{M\to\infty} \left(\varepsilon_{3,M}^{(i)} + \bar{\varepsilon}_{3,M}^{(i)}\right)$ while $E_3 = E_3^{(1)} = E_3^{(2)}$

# Descendent states and $q_3 = 0$ states for $N = 4$

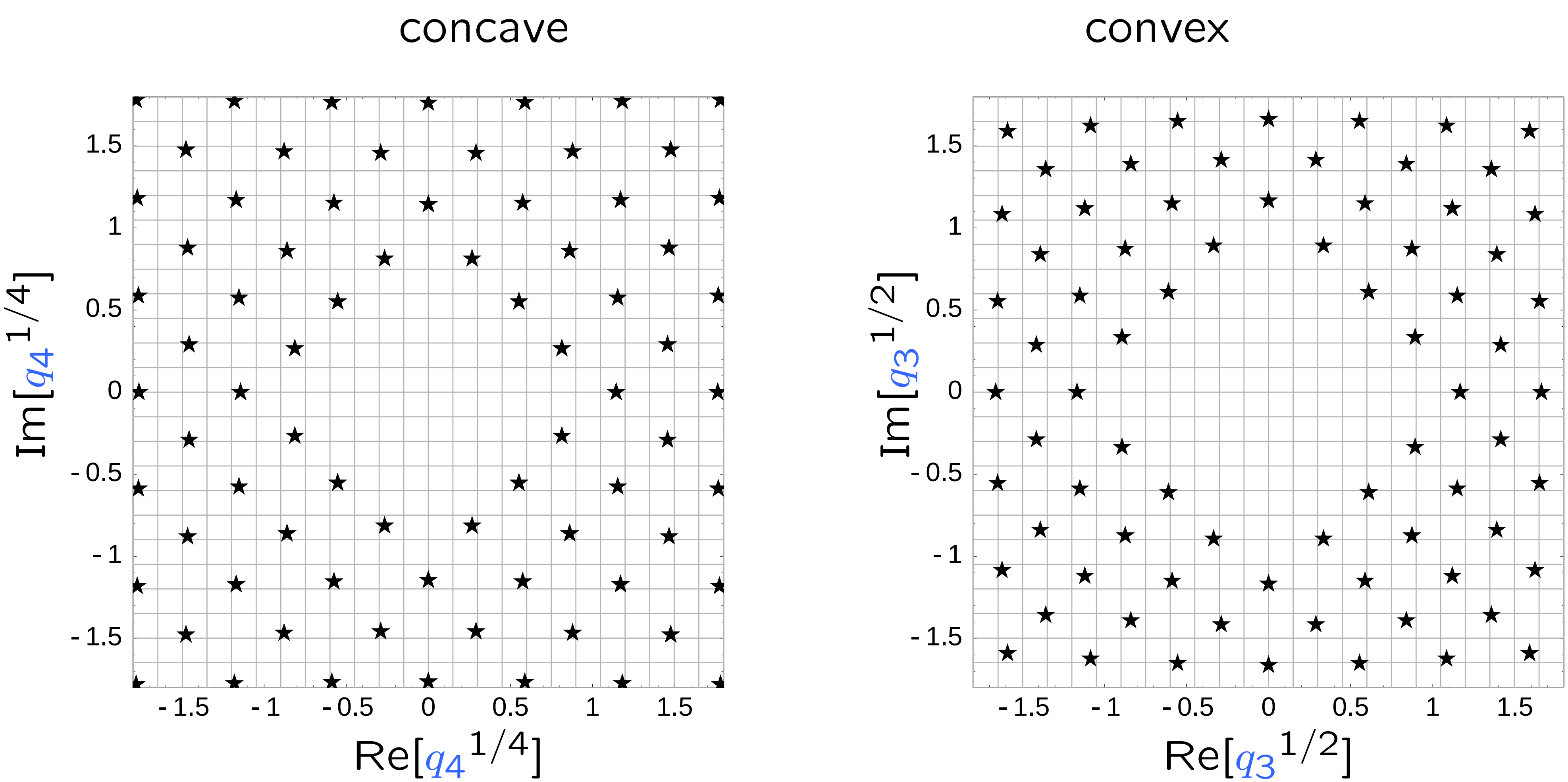
Descendent states:
$q_4 = 0$ and $h = 1/2$



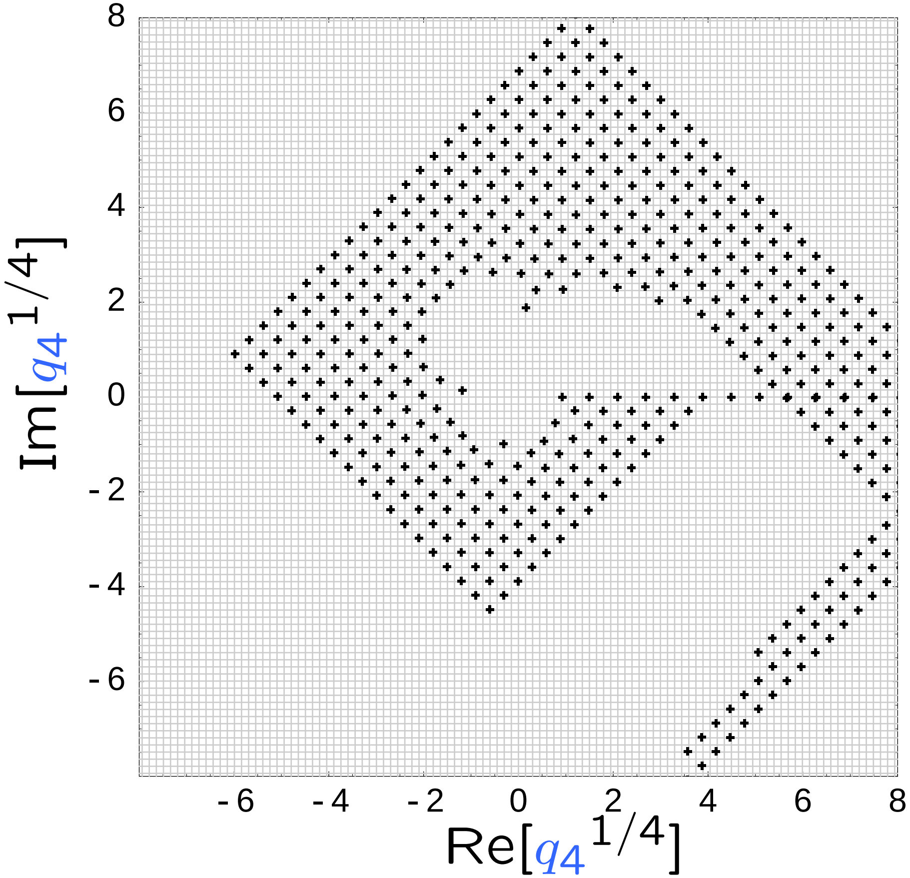

States with
$q_3 = 0$ and $h = 1/2$

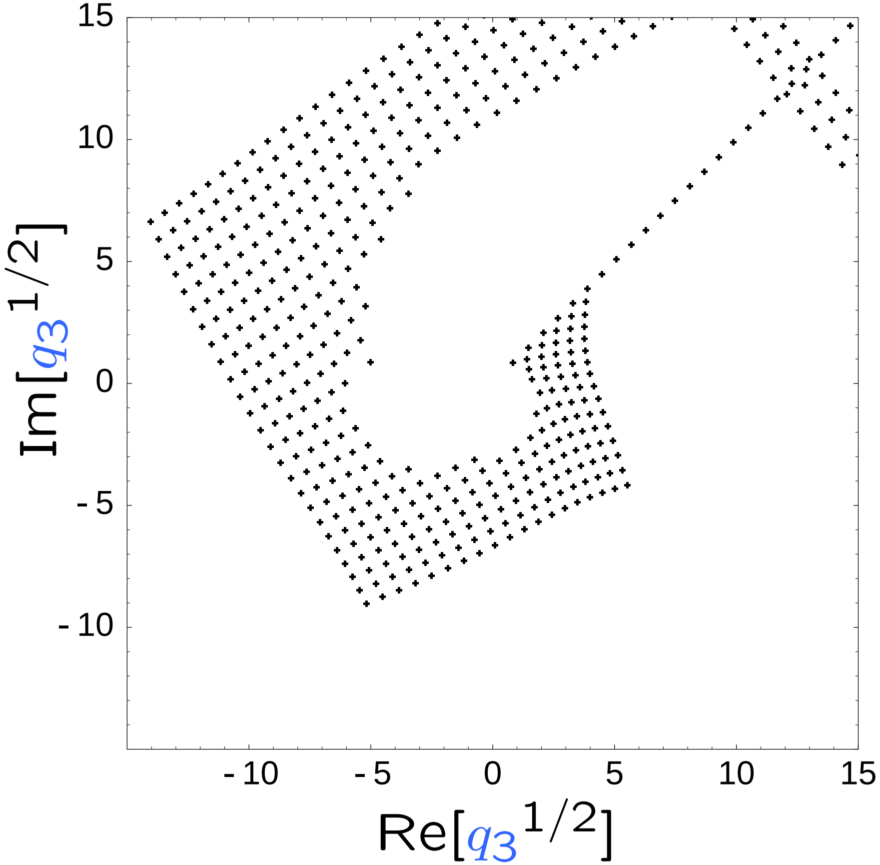


# Resemblant lattices for $N = 4$ and $h = 1/2$



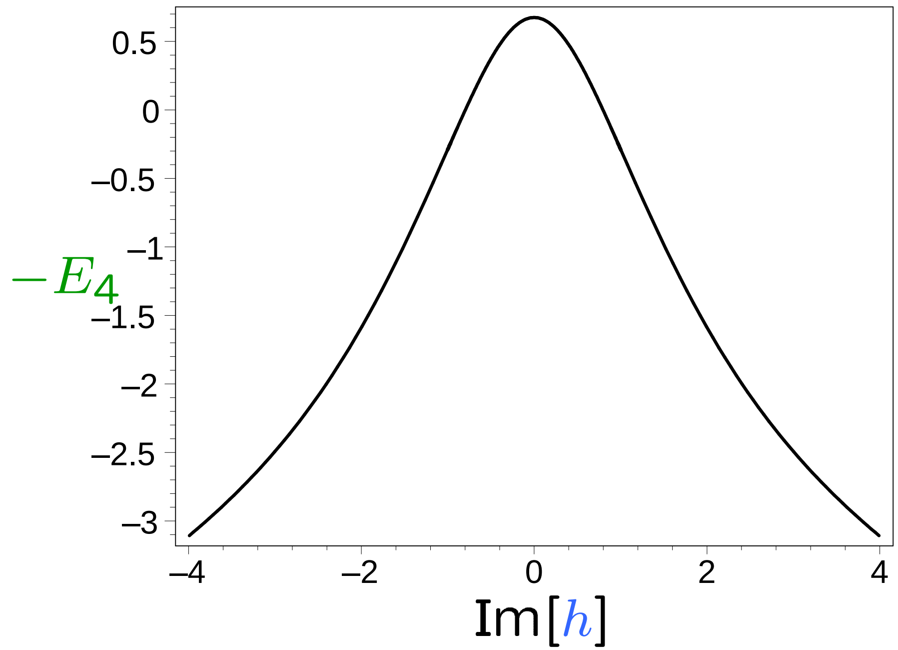
# Winding lattices for $N = 4$ and $h = 1/2$



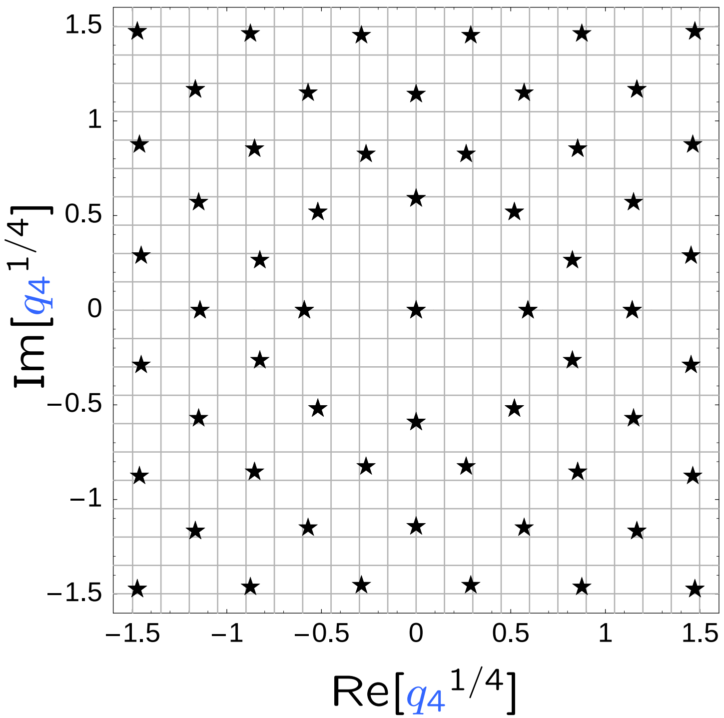

Here there are only some vertices

# Energy for $N = 4$ and $h = 1/2$




| $(\ell_1/2, \ell_2/2)$ | $\left(q_4{}^{\text{exact}}\right)^{1/4}$ | $\left(q_4{}^{\text{WKB}}\right)^{1/4}$ | $-E_4$ |
|---|---|---|---|
| $(2,0)$ | $0.626$ | $0.599$ | $0.6742$ |
| $(2,2)$ | $0.520 + 0.520\,i$ | $0.599 + 0.599\,i$ | $-1.3783$ |
| $(3,1)$ | $0.847 + 0.268\,i$ | $0.899 + 0.299\,i$ | $-1.7919$ |
| $(4,0)$ | $1.158$ | $1.198$ | $-2.8356$ |
| $(3,3)$ | $0.860 + 0.860\,i$ | $0.899 + 0.899\,i$ | $-3.1410$ |
| $(4,2)$ | $1.159 + 0.574\,i$ | $1.198 + 0.599\,i$ | $-3.3487$ |

# Non-trajectory lattice for $N = 4$ and $h = 1$




| $(\frac{\ell_1}{2}, \frac{\ell_2}{2})$ | $(q_4{}^{\text{exact}})^{1/4}$ | $(q_4{}^{\text{WKB}})^{1/4}$ | $-E_4$ |
|---|---|---|---|
| $(2, 0)$ | $0.591$ | $0.599$ | $1.03996$ |
| $(0, 0)$ | $0.$ | $0.$ | $0.$ |
| $(2, 2)$ | $0.519 + 0.519\,i$ | $0.599 + 0.599\,i$ | $-0.29054$ |
| $(3, 1)$ | $0.826 + 0.264\,i$ | $0.899 + 0.299\,i$ | $-1.72366$ |
| $(4, 0)$ | $1.142$ | $1.198$ | $-2.79597$ |
| $(3, 3)$ | $0.854 + 0.854\,i$ | $0.899 + 0.899\,i$ | $-3.09125$ |
| $(4, 2)$ | $1.149 + 0.571\,i$ | $1.198 + 0.599\,i$ | $-3.31240$ |

Lipatov's ground state for $N = 4$: $(2, 0)$

Point-like lattice for $h = 1$; no continuity in $\nu_h$ $(h = \frac{1+n_h}{2} + i\,\nu_h)$

# QCD Hamiltonian in the Multi-Colour Limit: $N_c \to \infty$

$$\mathcal{H}_N = \sum_{k=0}^{N-1} H(\vec{z}_k, \vec{z}_{k+1})$$

where $\vec{z}_0 \equiv \vec{z}_N$

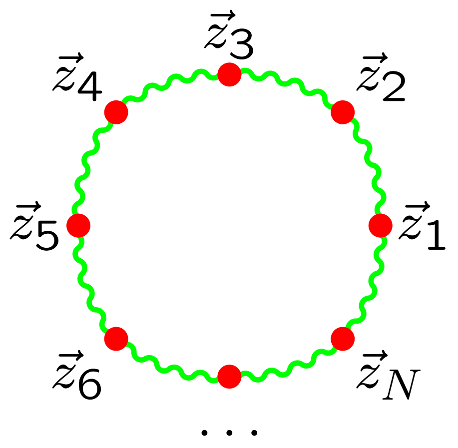
- holomorphic and antiholomorphic coordinates

$$\vec{z}_k \equiv (x_k, y_k) \leftrightarrow \begin{cases} z_k = x_k + iy_k \\ \overline{z}_k = x_k - iy_k \end{cases}$$

- Bose symmetry $\longrightarrow$ invariant under cyclic and mirror permutations of particles

$$\mathbb{P}\Psi(\vec{z}_1, \ldots, \vec{z}_{N-1}, \vec{z}_N) = \Psi(\vec{z}_2, \ldots, \vec{z}_N, \vec{z}_1)$$

$$\mathbb{M}\Psi(\vec{z}_1, \ldots, \vec{z}_{N-1}, \vec{z}_N) = \Psi(\vec{z}_N, \ldots, \vec{z}_2, \vec{z}_1)$$



• The Schrödinger Equation defines a quantum-mechanical model of $N$ interacting particles on the cylinder-wrapped plane which can be identified as a two dimensional non-compact XXX Heisenberg $SL(2,\mathbb{C})$ spin chain model.

The number of sites of the model is equal to the number of Reggeons $N$ an the spins attached to each $k$-site $S_k^{\pm,0}$, $\overline{S}_k^{\pm,0}$

$$\tilde{S}_k = \begin{cases} S_k^0 = z_k \partial_{z_k} + j \\ S_k^- = -\partial_{z_k} \\ S_k^+ = z_k^2 \partial_{z_k} + 2 j z_k \end{cases} \qquad \text{similarly for } \tilde{\overline{S}}_k = \left\{ \overline{S}_k^{\pm,0} \right\} \quad \text{with } z_k, j \to \overline{z}_k, \overline{j}$$

where $(j, \overline{j})$ are complex spins $\overline{j} = 1 - j^*$ (in QCD $j = 0$, $\overline{j} = 1$)

$$\boxed{\mathcal{H}_N = \sum_{k=1}^{N} \left[ H(J_{k,k+1}) + H(\overline{J}_{k,k+1}) \right]}$$

where $H(J) = \Psi(1 - J) + \Psi(J) - 2\Psi(1)$, $\qquad \Psi(x) = \frac{d \ln \Gamma(x)}{dx}$

and $J_{k,k+1}(J_{k,k+1} - 1) = (\tilde{S}_k + \tilde{S}_{k+1})^2$, $\qquad J_{N,N+1} = J_{N,1}$

- possesses the complete set of the conserved charges
  (integrals of motion) – invariant products of conformal spins

$$\left[\mathcal{H}_N, \widehat{\vec{q}}_n\right] = \left[\widehat{\vec{q}}_n, \widehat{\vec{q}}_m\right] = 0 \qquad\qquad n, m = 2, \ldots, N$$

e.g.: $q_2 = \sum_{\{j,k\}} \widetilde{S}_j \cdot \widetilde{S}_k$, $\qquad q_3 = \sum_{\{j,k,l\}} \widetilde{S}_j \cdot \left(\widetilde{S}_k \times \widetilde{S}_l\right)$

Eigenvalues of the lowest conformal charge can be written as

$$q_2 = -h(h-1) + Nj(j-1) \qquad \text{and } \bar{q}_2 = q_2^*$$

where $h = 1 + \frac{n_h}{2} + i\nu_h \qquad \bar{h} = 1 - h^* \qquad$ and $n_h \in \mathbb{Z}$ $\nu_h \in \mathbb{R}$

Eigenvalues of $\mathbb{P}$

$$\mathbb{P}\Psi_{\vec{q}}(\{\vec{z}_i\}) = \exp(i\theta_N(\vec{q}))\Psi_{\vec{q}}(\{\vec{z}_i\}), \qquad \mathbb{P}^N = 1$$

$$\mathbb{M}\Psi_{\vec{q}}(\{\vec{z}_i\}) = \Psi_{-\vec{q}}(\{\vec{z}_i\}) \qquad \text{where } \vec{q} = \{\vec{q}_k\} \text{ and } -\vec{q} = \{(-1)^k \vec{q}_k\}$$

$$\boxed{\text{the quasimomentum: } \theta_N(\vec{q}) = 2\pi k/N,\ k = 0, 1, \ldots, N-1}$$

# The Baxter $Q$-Operator: $\widehat{\mathbb{Q}}_{\vec{q}}(u, \overline{u})$

Hamiltonian:

$$\boxed{\begin{aligned}\mathcal{H}_N(\vec{q}) = i\frac{d}{du} \ln \Big\{ u^{2N} \Big[ & \widehat{\mathbb{Q}}_{\vec{q}}(u - i(1-j), u - i(1-\overline{j})) \\ & \times \widehat{\mathbb{Q}}_{\vec{q}}(u + i(1-j), u + i(1-\overline{j})) \Big] \Big\} \Big|_{u=0}\end{aligned}}$$

- $\widehat{\mathbb{Q}}_{\vec{q}}(u, \overline{u})$ Commutes:

$$\left[\widehat{\mathbb{Q}}_{\vec{q}}(u, \overline{u}), \widehat{\mathbb{Q}}_{\vec{q}}(v, \overline{v})\right] = \left[\hat{t}_N(u), \widehat{\mathbb{Q}}_{\vec{q}}(v, \overline{v})\right] = \left[\hat{\overline{t}}_N(\overline{u}), \widehat{\mathbb{Q}}_{\vec{q}}(v, \overline{v})\right] = 0,$$

$$\hat{t}_N(u) = 2u^N + \hat{q}_2 u^{N-1} + \ldots + \hat{q}_N$$

where $u$, $\overline{u}$: $2i(u - \overline{u}) \in \mathbb{Z}$

- Its eigenvalues satisfy the Baxter equations:

$$\begin{cases} \hat{t}_N(u)\mathbb{Q}(u, \overline{u}) = (u + ij)^N \mathbb{Q}(u + i, \overline{u}) + (u - ij)^N \mathbb{Q}(u - i, \overline{u}) \\ \hat{\overline{t}}_N(\overline{u})\mathbb{Q}(u, \overline{u}) = (\overline{u} + i\overline{j})^N \mathbb{Q}(u, \overline{u} + i) + (\overline{u} - i\overline{j})^N \mathbb{Q}(u, \overline{u} - i) \end{cases}$$

- One have prescribed asymptotic behaviour at infinity and analytical properties – known pole structure:

$$u^{\pm}_{m} = \pm i(j - m), \ \overline{u}^{\pm}_{\overline{m}} = \pm i(\overline{j} - \overline{m}) \qquad m, \overline{m} \in \text{positive integers}$$

$$\mathbb{Q}(u^{\pm}_{1} + \varepsilon, u^{\pm}_{1} + \varepsilon) = R^{\pm}(\vec{q}) \left[ \frac{1}{\varepsilon^{N}} + \frac{iE^{\pm}(\vec{q})}{\varepsilon^{N-1}} + \dots \right]$$

$$\boxed{E_N(\vec{q}) = \mathrm{Re}\left[E^{+}(-\vec{q}) + E^{+}(\vec{q})\right]} \qquad \boxed{\theta_N(\vec{q}) = 2i \ln \frac{R^{+}(-\vec{q})}{R^{+}(\vec{q})}}$$

# Solution of the Baxter Equation

Ansatz: $\mathbb{Q}(u, \overline{u}) = \int d^2 z z^{-iu-1} \overline{z}^{-i\overline{u}-1} Q_{\vec{q}}(z, \overline{z})$

leads to following differential equation on $Q_{\vec{q}}(z, \overline{z})$

$$\left[z^{j}(z\partial_z)z^{1-j} + z^{-j}(z\partial_z)^{N} z^{j-1} - 2(z\partial_z)^{N}\right] Q_{\vec{q}}(z, \overline{z}) = \sum_{k=2}^{N} q_k (z\partial_z)^{N-k} Q_{\vec{q}}(z, \overline{z})$$

$\overline{z}$ dependence of $Q_{\vec{q}}(z, \overline{z})$ is fixed by similar equation

in the $\overline{z}$ sector with $\overline{j}$, $\overline{q_k}$

Solution of this equations can be constructed as:

$$Q_{\vec{q}}(z,\overline{z}) = \sum_{n,\overline{n}=1}^{N} Q_n(z;q)C_{n\overline{n}}(\vec{q})\overline{Q}_{\overline{n}}(\overline{z};\overline{q})$$

The differential equations has three regular singular points :

$z = 0$, $z = 1$, $z = \infty$.

so solutions around these points are

$$\left.\begin{matrix} Q_k^{(0)}(z;q), & \overline{Q}_k^{(0)}(\overline{z};\overline{q}) \\ Q_k^{(1)}(z;q), & \overline{Q}_k^{(1)}(\overline{z};\overline{q}) \\ Q_k^{(\infty)}(z;q), & \overline{Q}_k^{(\infty)}(\overline{z};\overline{q}) \end{matrix}\right\} \overset{C_{n\overline{n}}^{(\dots)}(\vec{q})}{\Longleftrightarrow} \left\{\begin{matrix} Q_{\vec{q}}^{(0)}(z,\overline{z}) \\ Q_{\vec{q}}^{(1)}(z,\overline{z}) \\ Q_{\vec{q}}^{(\infty)}(z,\overline{z}) \end{matrix}\right.$$

contain logarithm and power functions of $z$, $\overline{z}$.

Should be single valued

To satisfy this requirement $C_{n\overline{n}}^{(\dots)}(\vec{q})$ has to have proper structure

i.e.: some elements of it should vanished

Transition matrices are defined by:

$$\begin{cases} Q_m^{(0)}(z;q) = \displaystyle\sum_{n=1}^{N} \Omega_{mn}(q) Q_n^{(1)}(z;q) \\ Q_{\overline{m}}^{(0)}(\overline{z};\overline{q}) = \displaystyle\sum_{\overline{n}=1}^{N} \overline{\Omega}_{\overline{mn}}(\overline{q}) Q_{\overline{n}}^{(1)}(\overline{z};\overline{q}) \end{cases}$$

$$Q_{\vec{q}}^{(1)}(z,\overline{z}) = \textstyle\sum_{n,\overline{n}=1}^{N} Q_n^{(1)}(z;q) C_{n\overline{n}}^{(1)}(\vec{q}) Q_{\overline{n}}^{(1)}(\overline{z};\overline{q})$$

$$\begin{aligned} Q_{\vec{q}}^{(0)}(z,\overline{z}) &= \textstyle\sum_{m,\overline{m}=1}^{N} Q_m^{(0)}(z;q) C_{m\overline{m}}^{(0)}(\vec{q}) Q_{\overline{m}}^{(0)}(\overline{z};\overline{q}) \\ &= \textstyle\sum_{n,\overline{n},m,\overline{m}=1}^{N} Q_n^{(1)}(z;q) \Omega_{mn}(q) C_{m\overline{m}}^{(0)}(\vec{q}) \overline{\Omega}_{\overline{mn}}(\overline{q}) Q_{\overline{n}}^{(1)}(\overline{z};\overline{q}) \end{aligned}$$

$$\Longrightarrow \boxed{C^{(1)}(\vec{q}) = [\Omega(q)]^T \cdot C^{(0)}(\vec{q}) \cdot \overline{\Omega}(\overline{q})}$$

| E$^{\text{qn}}$ | $C^{(0)}$ | $C^{(1)}$ | $\vec{q}$ | |
|---|---|---|---|---|
| $N^2$ | $-(N-1)$ | $-(2+(N-2)^2)$ | $-(N-2)$ | $=(2N-3)$ |

$\Rightarrow$ quantization conditions for $\vec{q}$