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Lattice and Effective Field Theory for Cold Fermionic Atoms

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QCD and cold atoms: common thread

Investigation of nonperturbative phenomena

**Effective Field Theory**
Theoretical separation of scales

+  

**Lattice Calculation**
Monte Carlo evaluation of path integral

=  **Rigorous Nonperturbative Results**
Cold Fermionic Atoms - Outline

- General motivation, specific system
- Lattice Field Theory & Monte Carlo calculation
  - First results -- superfluid/normal phase transition
  - Road to understanding and reducing uncertainties
  - (see M.W., cond-mat/0502372)
- Symmetries & Low Energy Effective Field Theory
  - All cold atoms: general coordinate invariance
  - Unitary Fermi gas: scale and conformal invariance
- Future directions
Trapped Atoms are **Versatile!**

- **Atomic theory**
  - Feshbach resonances, few-body effects

- **Condensed matter theory**
  - Superfluidity, optical lattices and semiconductor physics, very pure systems, interesting phase diagrams

- **Nuclear theory**
  - Few body physics, fermion pairing, model of neutron matter

- **Quantum field theory**
  - Separation of scales (good for effective field theory), spontaneous symmetry breaking, *interesting nonperturbative regime*, *universality*
The system

Homogeneous gas of 2 identical species of NR fermions

Hamiltonian

\[
H = -\frac{1}{2m} \left( \sum_{i=1}^{N_1} \nabla_i^2 + \sum_{j=1}^{N_2} \nabla_j^2 \right) + \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} v(|\vec{r}_i - \vec{r}_j|)
\]

Low energy scattering, short range potential

\[
\mathcal{A} = \frac{1}{-1/a + \frac{1}{2} k^2 R + \ldots - ik}
\]

Dilute limit \( R \ll n^{-1/3} \quad (k_F = (3\pi^2 n)^{1/3}) \)

Relevant physics can be described by scattering length

Strongly interacting \( n|a|^3 \gg 1 \quad \text{“Bertsch problem”} \)
Feshbach resonance

$$V(r)$$

closed channel, virtual bound state

van der Waals attraction, assume no bound state

low energy scattering
Feshbach resonance

\[ a = a_{bg} \left( 1 - \frac{\Gamma}{B - B_0} \right) \]

- van der Waals attraction, assume no bound state
- closed channel, virtual bound state
- vary \( \vec{B} \) field
- low energy scattering

\( V(r) \)
Fermion pairing at zero temperature

BCS

− 0 +

... ? ? ? ? ...

BEC

Illustration: A. Stonebraker (Science)
Finite temperature phase transition

High temperature = normal matter

Low temperature = superfluid matter

\[ T_c / T_F \]

\[ 1/(n^{1/3} a) \]

BCS

BEC
Relevance to nuclear physics

- Many-fermion systems
- Large scattering lengths in $NN$

\[
\ell_\pi = \frac{\hbar}{m_\pi c} = 1.4 \text{ fm}
\]

\begin{align*}
\text{nn} & \quad r_s = 2.8 \text{ fm} \quad a_s = -18.5 \text{ fm} \\
\text{np} & \quad ^1S_0 \quad r_s = 2.75 \text{ fm} \quad a_s = -23.76 \text{ fm} \\
& \quad ^3S_1 \quad r_t = 1.76 \text{ fm} \quad a_t = 5.42 \text{ fm}
\end{align*}
Physical vs. lattice length scales

The continuum limit is the dilute limit, \( nb^3 \to 0 \)

Also worry about finite volume \( b \ll n^{-1/3} \ll L \)
Atomic trap = inhomogeneous matter

Harmonic trap potential

\[ V(\vec{r}) = \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) \]

Introduces another physical length scale

\[ a_{ho} = \frac{1}{\sqrt{m\bar{\omega}}} \]

\[ b \ll n^{-1/3} \ll a_{ho} \ll L \quad \text{Difficult!} \]

\[ (\bar{\omega} \equiv (\omega_x \omega_y \omega_z)^{1/3}) \]

illustration: D. Jin’s group (JILA) website (Denver Post?)
Lattice field theory formulation

Write partition function as a path integral

\[ Z = \text{Tr} \exp[-\beta(\hat{H} - \mu \hat{N})] \]
\[ = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int_0^\beta dx_4 \int d^3x \, \mathcal{L}\right) \quad \psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]
finite 4th dimension (imaginary time)

\[ \mathcal{L} = \bar{\psi} \left( \partial_t - \frac{1}{2} \nabla^2 - \mu \right) \psi + \frac{C_0}{2} (\bar{\psi}\psi)^2 \]

U(1) symmetry: fermion number conservation

\[ \psi \rightarrow \exp(i\alpha)\psi \, , \quad \bar{\psi} \rightarrow \bar{\psi} \exp(-i\alpha) \]

Pair condensation: spontaneous symmetry breaking

\[ \langle (\psi^T \sigma_2 \psi + \bar{\psi} \sigma_2 \bar{\psi}^T) + e^{i\theta} (\psi^T \sigma_2 \psi - \bar{\psi} \sigma_2 \bar{\psi}^T) \rangle \neq 0 \]
Tuning of lattice parameters

- lattice chem. pot.: $\mu$
- interaction strength: $m^2 (= -C_0^{-1})$
- imaginary time extent: $\xi, N_t$

Monte Carlo

\[(V \to \infty, J \to 0)\]

fermion number density $n$

temperature $T$

2-body scattering length $a$

\[
\frac{m^2}{\xi} = -\frac{b_s}{4\pi a} + \int_{\text{BZ}} \frac{d^3 p}{(2\pi)^3} \left( \frac{1}{|\hat{p}|^2 (1 + |\hat{p}|^2 / 4\xi)} \right)
\]
Simulation details

- Spatial volume
  \[ V = (8b_s)^3 \]
- Imaginary time extent
  \[ N_t = 16 \]
- Fixed chemical potential (so far)
  \[ \mu b_t = 0.4 \Rightarrow nb_s^3 \approx 0.2 - 0.25 \]
- Simulations at several values of
  \( (m^2, \xi) \)
- 3 values of \( J \) (to extrapolate to \( J = 0 \))

- Each parameter set, 4000 Monte Carlo steps
  - 200 independent field configurations (toss first 50)
Extrapolation to zero external source

\[ \Sigma = \frac{1}{2} \langle \psi^T \sigma_2 \psi + \bar{\psi} \sigma_2 \bar{\psi}^T \rangle \]

\( \mu = 0.4, \ m^2 = 0.175 \)
Phase transition

Anisotropy (≈ Temperature)

Order parameter $\langle J \rangle$

$\Sigma(J\neq 0)$

$0.9$ $1$ $1.1$ $1.2$ $1.3$ $1.4$ $1.5$

$0$ $0.02$ $0.04$ $0.06$ $0.08$ $0.1$

$m^2 = 0.14$

$m^2 = 0.1456$

$m^2 = 0.155$

$m^2 = 0.175$
Exploratory results for critical temperature

M.W., arXiv:cond-mat/0502372

To do:
1) Phase transition across (constant scattering length)
2) Dilute (i.e. continuum) limit
3) Comparison of algorithms

High temperature = normal matter
Low temperature = superfluid matter

BCS

BEC
Other Monte Carlo calculations of $T_c$

- **Auxilliary field M.C.** (Bulgac, Drut, & Magierski, cm/0505374)
  - Look for shoulders in $E$ and $S$ vs. $T$
  - $T_c/T_F = 0.23(2)$

- **Hybrid M.C.** (Lee & Schäfer, cm/0509018)
  - $T_c/T_F < 0.14$

- **Truncated determinant diagrammatic Monte Carlo**
  (Bourovski, et al., cm/0602224)
  - Finite volume analysis, continuum limit
  - $T_c/T_F = 0.152(7)$
At very low temperatures, the only excitations are *phonons*

\[
\sqrt{\frac{2\pi}{mT}} \gg \frac{v_F}{\Delta_0}
\]

Leading order behavior: superfluid hydrodynamics (in bulk)
Thomas-Fermi theory (in traps)

Use *symmetries* of fermion Lagrangian to construct phonon effective Lagrangian, *beyond leading order*

Predictions for experiment *and Monte Carlo calculations*
Introducing external sources

\[ \mathcal{L} = \frac{i}{2} \psi^\dagger \overleftrightarrow{D_t} \psi - \frac{g^{ij}}{2m} D_i \psi^\dagger D_j \psi + q_0 \psi^\dagger \psi \sigma - \frac{1}{2} g^{ij} \partial_i \sigma \partial_j \sigma - \frac{\sigma^2}{2r_0^2} \]

U(1) gauge field \( D_\mu \psi = (\partial_\mu + i A_\mu) \psi \)  
3D metric tensor \( g^{ij} \)

\[ \exp\{iW[A_0, A_i, g_{ij}]\} = \int D\psi D\psi^\dagger \exp\{iS[\psi, \psi^\dagger, A_0, A_i, g_{ij}]\} \]

number density \( n = -\frac{\delta W}{\delta A_0} \)
number current \( j^k = -\frac{\delta W}{\delta A_k} \)
momentum density \( T_{0k} = -mg_{ik} \frac{\delta W}{\delta A_i} + A_k \frac{\delta W}{\delta A_0} \)
stress tensor \( T^i_k = 2g_{kj} \frac{\delta W}{\delta g_{ij}} - \delta^i_k W + A_k \frac{\delta W}{\delta A_i} \)
Arguments for validity of this microscopic theory

- Gauge field, spatial curvature are sources for fermion number density, current density, and stress tensor -- spurion analysis

- Nonrelativistic limit of relativistic theory

Hypothesize this is correct microscopic Lagrangian

- Testable hypothesis, has observable consequences
Symmetry breaking and the phonon

U(1) symmetry, conservation of particle number

\[ \psi \rightarrow e^{i\alpha} \psi, \quad \psi^\dagger \rightarrow \psi^\dagger e^{-i\alpha} \]

Superfluidity from spontaneous symmetry breaking

\[ \langle \psi \psi \rangle = |\langle \psi \psi \rangle| e^{-2i\phi} \neq 0 \]

\( \phi(t, \vec{x}) \) is the corresponding Goldstone mode, the phonon

Absorb chemical potential

\[ \theta(t, \vec{x}) \equiv \mu t - \phi(t, \vec{x}) \]
General coordinate invariance

\[ x^i \rightarrow x^i + \xi^i(t, \vec{x}) \]

\[
\begin{align*}
\delta \psi &= i \alpha \psi - \xi^k \partial_k \psi \\
\delta A_0 &= -\dot{\alpha} - \xi^k \partial_k A_0 - A_k \dot{\xi}^k \\
\delta A_i &= -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + mg_{ik} \dot{\xi}^k \\
\delta g_{ij} &= -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{kj} \partial_i \xi^k \\
\delta \theta &= \alpha - \xi^k \partial_k \theta \\
\delta (D_t \theta) &= -\xi^k \partial_k D_t \theta - \dot{\xi}^k D_k \theta \\
\delta (g^{ij} D_i \theta D_j \theta) &= -\xi^k \partial_k (g^{ij} D_i \theta D_j \theta) - 2m \dot{\xi}^k D_k \theta
\end{align*}
\]

\[ L^{(0)}_{\text{eff}} = P(X) \quad X \equiv D_t \theta - \frac{g^{ij}}{2m} D_i \theta D_j \theta \]

Note: GCI holds for cold atoms in general!
Galilean invariance

... is a special case of general coordinate invariance

\[ g^{ij} = \delta^{ij} \quad \alpha = m\vec{v} \cdot \vec{x} \quad \xi^k = v^k t \]

\[
\psi(t, \vec{x}) \rightarrow \psi'(t, \vec{x}) = e^{im\vec{v} \cdot \vec{x}} \psi(t, \vec{x} - \vec{v}t) \\
A_0(t, \vec{x}) \rightarrow A'_0(t, \vec{x}) = A_0(t, \vec{x} - \vec{v}t) - v^k A_k(t, \vec{x} - \vec{v}t) \\
A_i(t, \vec{x}) \rightarrow A'_i(t, \vec{x}) = A_i(t, \vec{x} - \vec{v}t)
\]

We will find that at NLO, general coordinate invariance is \textbf{more restrictive} than Galilean invariance
Leading order phonon Lagrangian

\[ \mathcal{L}_{\text{eff}}^{(0)} = P(X) \quad X \equiv D_t \theta - \frac{g^{ij}}{2m} D_i \theta D_j \theta \]

\( P(X) \) is the same function as the pressure \( P(\mu) \)

\[ n = -\frac{\partial \mathcal{L}}{\partial A_0} = -\frac{\partial \mathcal{L}}{\partial X} \frac{\partial X}{\partial A_0} = \frac{\partial P}{\partial X} \quad n = \frac{\partial P}{\partial \mu} \]

Unitary Fermi gas: scale invariance and dimensional analysis

\[ P(X) = c_0 m^{3/2} X^{5/2} \]

Energy per particle

\[ \varepsilon(n) = \frac{3}{5} \left( \frac{2n}{5c_0} \right)^{2/3} n \quad \frac{n}{m} = \xi \frac{3}{5} \left( \frac{3\pi^2 n}{2m} \right)^{2/3} n \]
Phonon Lagrangian

General coordinate invariance yields the combination

\[ \mathcal{L} = P(X) \]

\[ X = D_t \theta - \frac{(D_i \theta)^2}{2m} \]

\[ + \partial_i X \partial_i X f_1(X) + (\partial_i D_i \theta)^2 f_2(X) \]

\[ + \left( -m \partial_i E_i + \frac{1}{4} F_{ij} F_{ij} - \partial_i F_{ij} D_j \theta \right) f_3(X) \]

\[ (\mathcal{L}_4 = R_{3D} f_4(X)) \]

Galilean invariant operators which don’t satisfy GCI

\[ F_{ij} F_{ij} \]

\[ m \partial_i E_i + \partial_i F_{ij} D_j \theta \]

\[ m^2 E_i^2 + 2m E_i F_{ik} D_k \theta + F_{ij} F_{ik} D_j \theta D_k \theta \]
Scale invariance

Unitary Fermi gas

\[ t \rightarrow t' = \lambda^{-1} t \]

\[ \vec{x} \rightarrow \vec{x}' = \lambda^{-1/2} \vec{x} \]

(really a coordinate transformation)

\[ f_1(X) = c_1 m^{1/2} X^{-1/2} \]
\[ f_2(X) = c_2 m^{-1/2} X^{1/2} \]
\[ f_3(X) = c_3 m^{-1/2} X^{1/2} \]

Functions become known, up to multiplicative constants
Conformal invariance

More general reparameterization of time

\[ t \rightarrow t' = t + \beta(t) \quad \text{scale transformation: } \beta(t) = bt \]

\[ \delta \psi = -\beta \dot{\psi} - \frac{3}{4} \dot{\beta} \psi \]
\[ \delta g_{ij} = -\beta \dot{g}_{ij} + \dot{\beta} g_{ij} \]
\[ \delta A_0 = -\beta \dot{A}_0 - \dot{\beta} A_0 \]
\[ \delta A_i = -\beta \dot{A}_i \]

Invariance of Yukawa-like interaction

\[ [\psi] = \frac{3}{4} = [\sigma] \quad [q_0] = \frac{1}{4} \quad \& \quad [r_0] = -\frac{1}{2} \quad \Rightarrow [q_0^2 r_0] = 0 \]

Further constrains NLO EFT

\[ c_3 = -9 c_2 \]
Phonon Lagrangian to NLO

General coordinate invariance yields the combination

\[ \mathcal{L} = P(X) \quad X = D_t \theta - \frac{(D_i \theta)^2}{2m} \]

\[ + \partial_i X \partial_i X f_1(X) + (\partial_i D_i \theta)^2 f_2(X) \]

\[ + \left( -m \partial_i E_i + \frac{1}{4} F_{ij} F_{ij} - \partial_i F_{ij} D_j \theta \right) f_3(X) \]

Conformal invariance tells us the functional form

\[ \mathcal{L} = c_0 m^{3/2} X^{5/2} + c_1 \sqrt{m} (\partial_i X \partial_i X) X^{-1/2} \]

\[ + \frac{c_2}{\sqrt{m}} \left[ (\partial_i D_i \theta)^2 + 9m \partial_i E_i - \frac{9}{4} F_{ij} F_{ij} + 9 \partial_i F_{ij} D_j \theta \right] X^{1/2} \]

all gases

unitary Fermi gas
Applying the EFT

Equation of state
\[ \frac{E}{N} = \xi \frac{E_{\text{free}}}{N} \quad c_0 = \frac{2^{5/2}}{15\pi^2 \xi^{3/2}} \]

Dispersion relation
\[ \omega(q) = \sqrt{\frac{\xi}{3}} v_F q \left[ 1 - \pi^2 \sqrt{2\xi} \left( c_1 + \frac{3}{2} c_2 \right) \frac{q^2}{k_F^2} \right] + O(q^5 \ln q) \]

Static density response function
\[ \chi(q) = -\frac{m k_F}{\pi^2 \xi} \left[ 1 + 2\pi^2 \sqrt{2\xi} \left( c_1 - \frac{9}{2} c_2 \right) \frac{q^2}{k_F^2} \right] + O(q^4 \ln q) \]

Static transverse response function
\[ \chi^T(q) = -9 c_2 \sqrt{\frac{\xi}{2}} v_F q^2 + O(q^4 \ln q) \]
Prediction

- Eliminate unknown constants to obtain a prediction

\[ \omega(q) = \sqrt{\frac{\xi}{3}} v_F q \left[ 1 - \frac{\chi(q) - \chi(0)}{2\chi(0)} + \frac{4\pi^2}{3} \frac{\chi^T(q)}{v_F k_F^2} \right] \]

- Not testable by experiment in near future

- Challenge for Monte Carlo methods

- Make more predictions
Current and future effort - atoms

- **Lattice field theory** -- Very sound theoretical approach
  - No ground state input necessary
  - No sign problem (therefore, no fixed-node approximation, etc.)
  - Errors are systematically reducible
  - More precision by determining lines of constant physics in lattice parameter space

- **Effective field theory**
  - Utilize all possible symmetries
  - Systematically include corrections to superfluid hydrodynamics and Thomas-Fermi theory
  - Finite volume, finite source \((J)\) EFT in progress w/ J.-W. Chen
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Relativistic Root of Gen. Coord. Inv.

Relativistic, free boson action in curved spacetime

\[ S = - \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_\mu \Psi^\ast \partial_\nu \Psi + m^2 c^2 \Psi^\ast \Psi) \]

Nonrelativistic limit \( c \to \infty \)

\[ \Psi = e^{-imc^2t} \frac{\psi}{\sqrt{2mc}} \]
\[ g_{\mu\nu} = \begin{pmatrix} -1 - \frac{2A_0}{mc^2} & -\frac{A_i}{mc} \\ -\frac{A_i}{mc} & g_{ij} \end{pmatrix} \]

\[ S = \int dt d^3x \sqrt{g} \left[ \frac{i}{2} \psi^\dagger \overset{\leftarrow}{\partial_t} \psi - A_0 \psi^\dagger \psi - \frac{g^{ij}}{2m} (\partial_i \psi^\dagger - iA_i \psi^\dagger)(\partial_j \psi + iA_j \psi) \right] \]

Nonrelativistic GC transformations follow from relativistic GCI
Loops enter at NNLO

Rescale field to obtain canonical normalization for K.E.

\[ \mathcal{L} \sim (\partial \varphi)^2 + \frac{\#}{\mu^2} (\partial_0 \varphi)(\partial_i \varphi)^2 + \frac{\#}{\mu^4} (\partial \phi)^4 + \cdots \]

Loops suppressed vs. tree by \[ \frac{p^4}{\mu^4} \ln(p/\mu) \]

vs. tree \[ p^2 \]

Loops suppressed vs. tree by \[ \frac{p^8}{\mu^8} \ln(p/\mu) \]

vs. tree \[ \frac{p^4}{\mu^4} \]