Using AdS/CFT to explore the strong coupling regime of gauge theories: II

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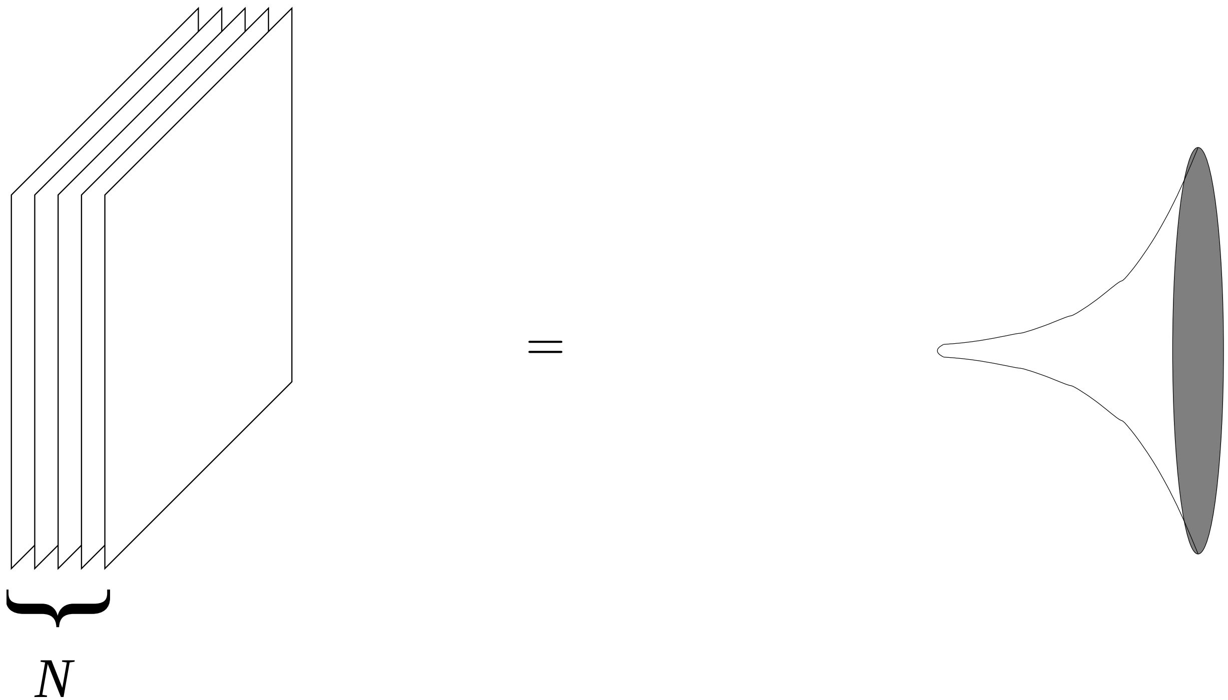
The Gauge/Gravity Duality

(Maldacena; Gubser, Klebanov, Polyakov; Witten)

Stack of N D3-branes in type IIB string theory: described in two different pictures:

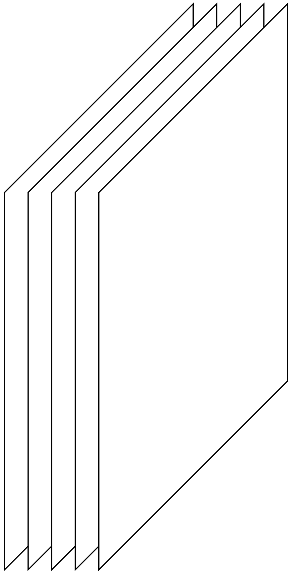
As a quantum field theory of degrees of freedom on the branes: a supersymmetric gauge theory

As string theory on a the curved spacetime (induced by the matter density on the branes)



The limit of infinitely strong coupling in gauge theory is the limit when string theory becomes Einstein's general relativity

Thermodynamics through AdS/CFT



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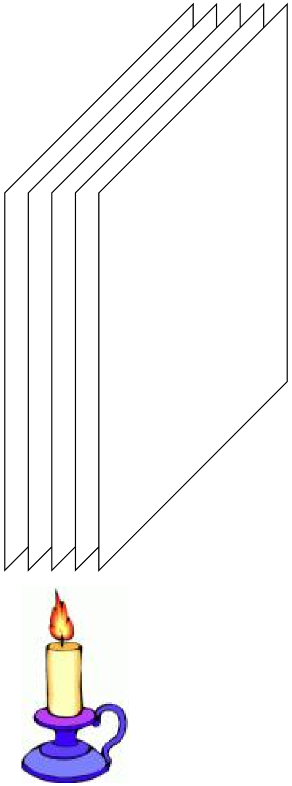

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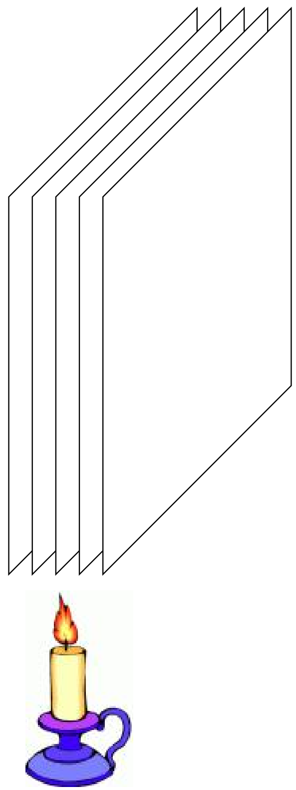

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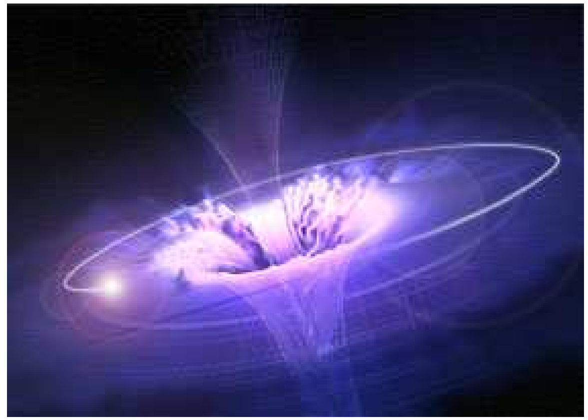
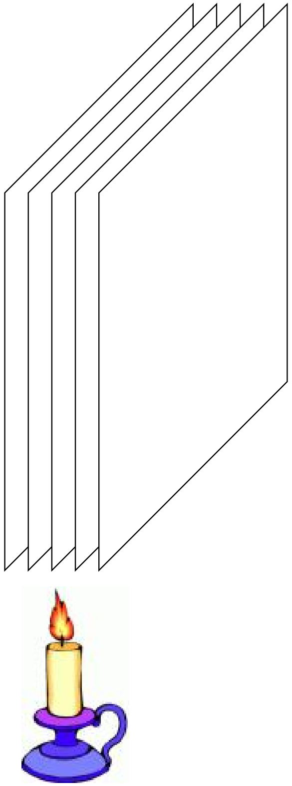


Thermal gauge theory = black hole in anti de-Sitter space

Thermodynamics through AdS/CFT



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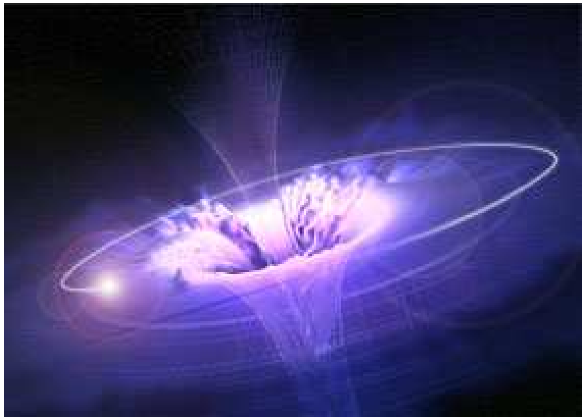


Thermal gauge theory = black hole in anti de-Sitter space

Consequence:
entropy of thermal gauge theory = Bekenstein-Hawking entropy of the black hole $\sim$ area of the event horizon. Factor $3/4$ for strongly coupled $\mathcal{N} = 4$ SYM theory.

AdS/CFT and viscosity

Idea: use gauge/gravity duality to investigate the hydrodynamic regime of field theory

finite-T QFT $\Leftrightarrow$black hole with translationally invariant horizon,

or “black brane”

Example (one among many): $\mathcal{N} = 4$ super-Yang-Mills theory at finite temperature

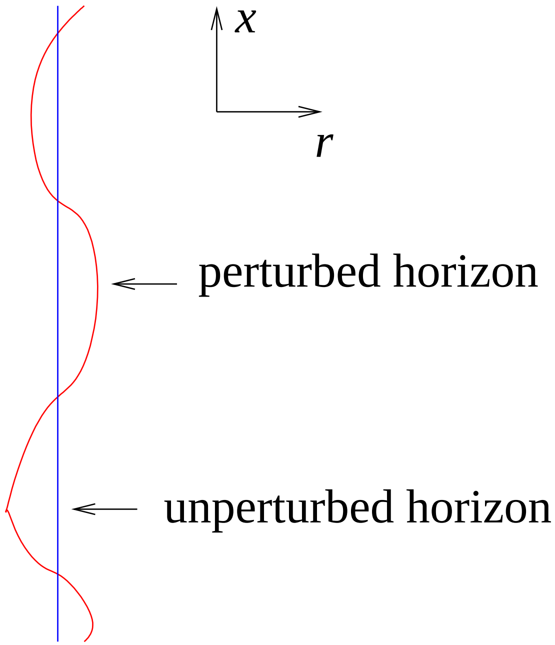
$$ds^2 = H^{-1/2}(-f dt^2 + dx^2 + dy^2 + dz^2) + H^{1/2}(\frac{dr^2}{f} + r^2 d\Omega_5^2)$$

where $H = 1 + R^4/r^4$, $f = 1 - r_0^4/r^4$, $r_0 \ll R$

$r = r_0$ is the location of horizon: flat in 3 dimensions x, y and z.

Hawking temperature $T = \frac{r_0}{\pi R^2}$

Dynamics of the horizon



$$T \sim r_0 = r_0(\mathbf{x})$$

Generalizing black hole thermodynamics M, Q,...
to black brane hydrodynamics

$$T = T_H(\mathbf{x}), \qquad \mu = \mu(\mathbf{x})$$

Event horizon behaves as a viscous fluid.

Viscosity: Kubo's formula

For our goal, it is best to start from Kubo's formula for viscosity:

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt\, d\mathbf{x}\, e^{i\omega t} \langle [T_{xy}(t, \mathbf{x}),\, T_{xy}(0, \mathbf{0})] \rangle$$

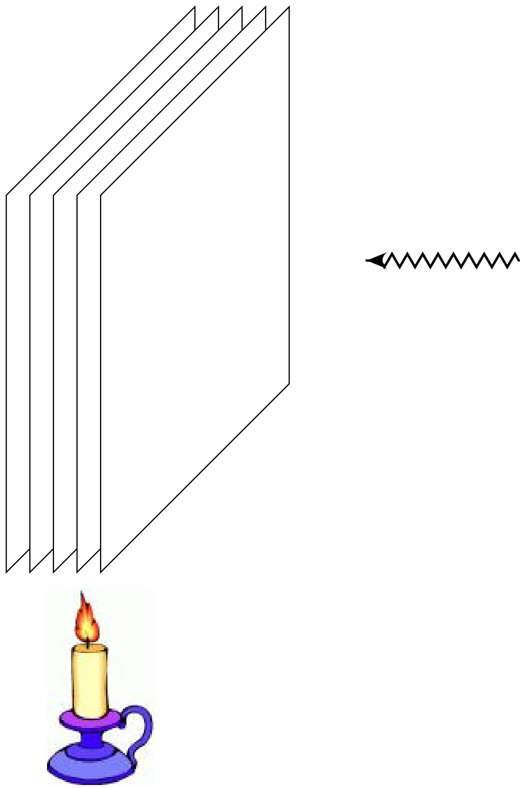
$$= - \lim_{\omega t \to 0} \lim_{\mathbf{q} \to 0} \mathrm{Im}\, G_{xy,xy}(\omega, \mathbf{q})$$

retarded Green's function of T_{xy}

Similar relations exist for other kinetic coefficients (diffusion constants, conductivities...)

Gravity counterpart of Kubo's formula

Consider a graviton that falls on this stack of N D3-branes
Will be absorbed by the D3 branes.
The process of absorption can be looked at from two different perspectives (Klebanov; Gubser, Klebanov, Tseytlin 1996-1997):



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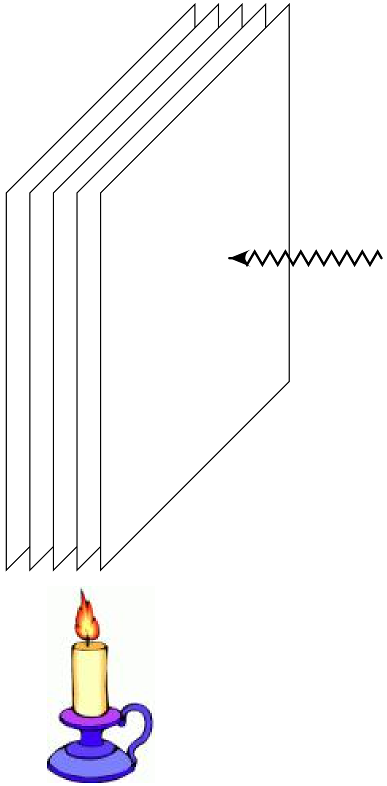

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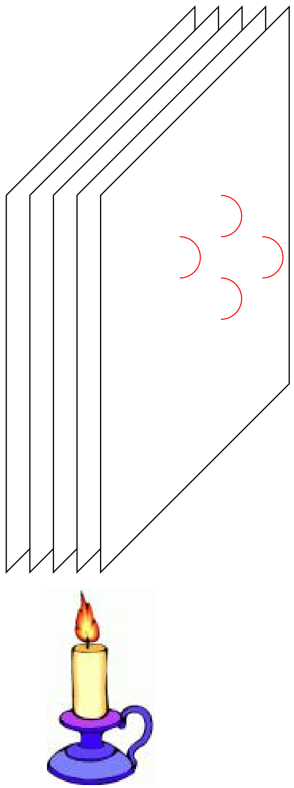

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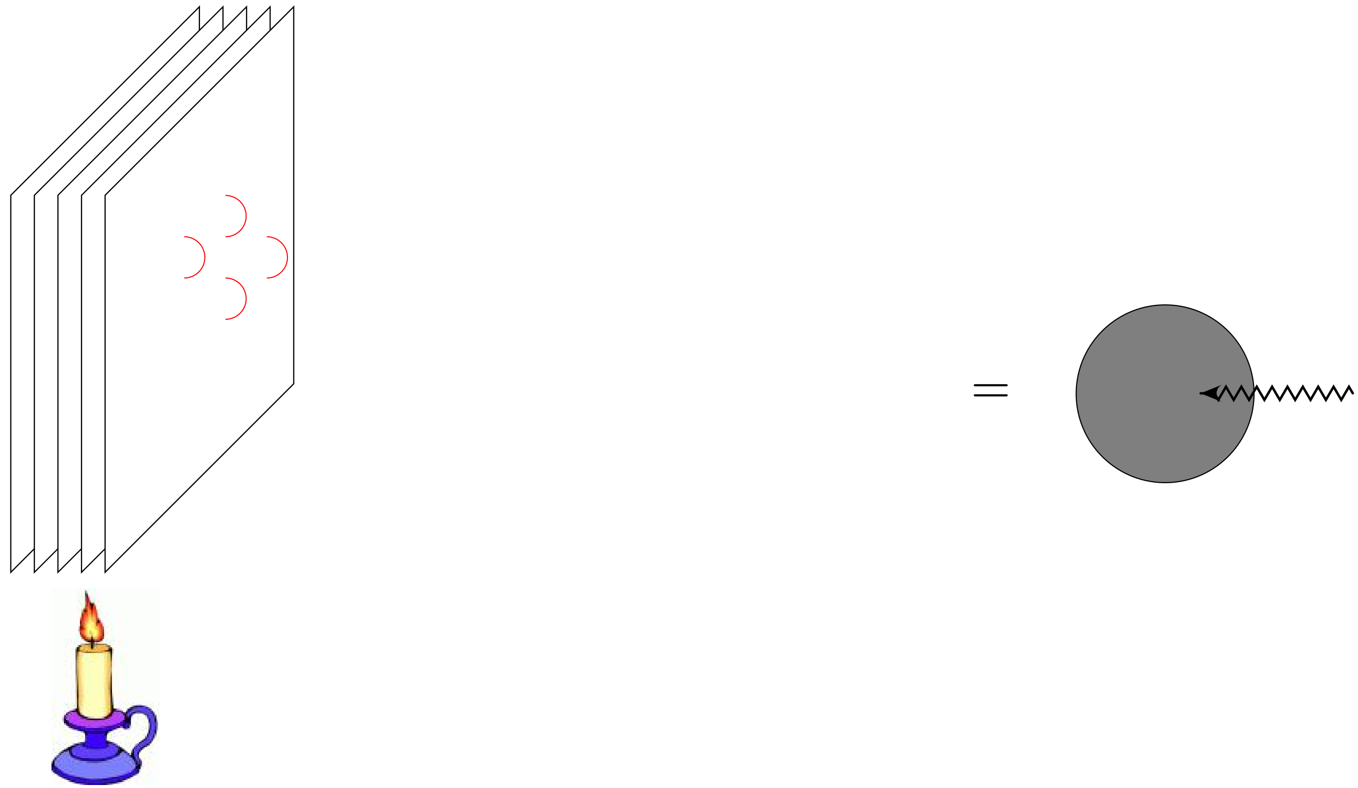


Absorption by D3 branes = absorption by black hole, calculable classically

Viscosity as absorption

From optical theorem: absorption cross section of graviton is proportional to the imaginary part of the correlator of stress-energy tensor of gauge theory (coupling: $h_{\mu\nu}T^{\mu\nu}$).

$$\sigma_{\text{abs}} = -\frac{2\kappa^2}{\omega}\,\mathrm{Im}\,G^R(\omega), \qquad \kappa = \sqrt{8\pi G}$$

$$= \frac{\kappa^2}{\omega}\int d^4x\, e^{i\omega t}\langle[T_{xy}(x),\, T_{xy}(0)]\rangle$$

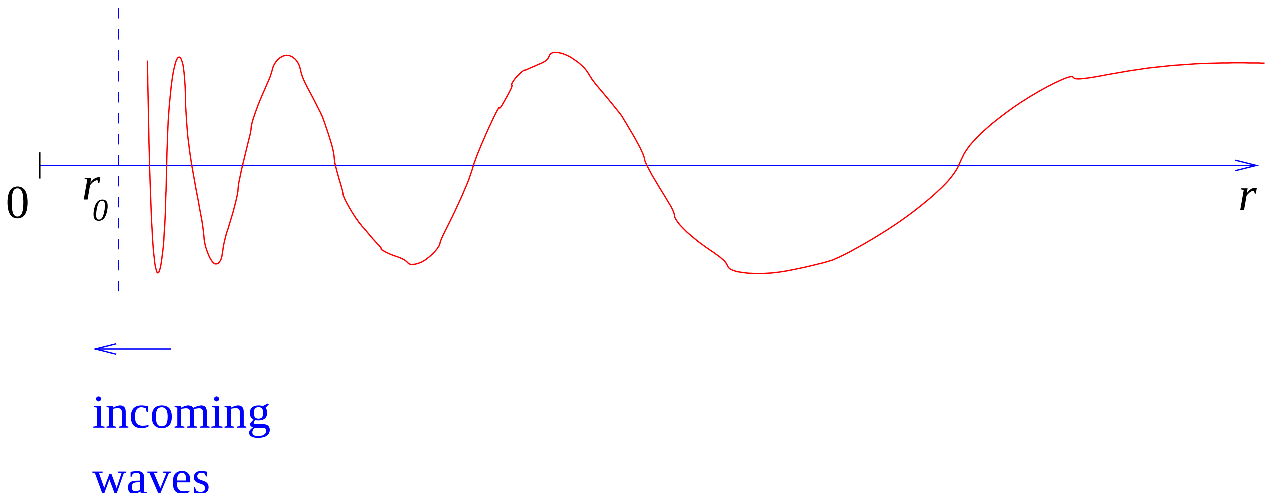
Viscosity = absorption cross section of low-energy gravitons

$$\eta = \frac{\sigma_{\text{abs}}(0)}{2\kappa^2} = \frac{\sigma_{\text{abs}}(0)}{16\pi G}$$

Absorption

Absorption cross section can be found classically by solving the wave equation "$\Box h_{xy}$"$= 0$

$$h''_{xy} + \frac{5r^4 - r_0^4}{r(r^4 - r_0^4)} h'_{xy} + \omega^2 \frac{r^4(r^4 + R^4)}{(r^4 - r_0^4)^2} h_{xy} = 0$$



Two theorems

The computation of the absorption cross section σ_{abs} is made easy by two theorems, valid for a wide class of backgrounds:

- Equation for h_{xy} is the same as of a minimally coupled scalar
- For a minimally coupled scalar

$$\lim_{\omega \to 0} \sigma_{\text{abs}} = \text{Area of the event horizon}$$

Das, Gibbons, Mathur

As a consequence of these theorems

$$\eta = \frac{\sigma_{\text{abs}}(0)}{16\pi G} = \frac{\text{Horizon area}}{16\pi G}$$

Universality of η/s

So we found

$$\eta = \frac{\text{Horizon area}}{16\pi G}$$

However the entropy density is

$$s = \frac{\text{Horizon area}}{4G}$$

Therefore

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

This result is valid for *all* theories with gravity duals: universal, but within a restrictive class of theories, including $\mathcal{N} = 4$ SYM theory at infinite 't Hooft coupling and deformations (less SUSY, non-conformal).
(also proven by Buchel and and Liu)

Restoring $\hbar$ and c

The dimensionality of η/s is the same as that of $\hbar$, in any number of spatial dimensions.

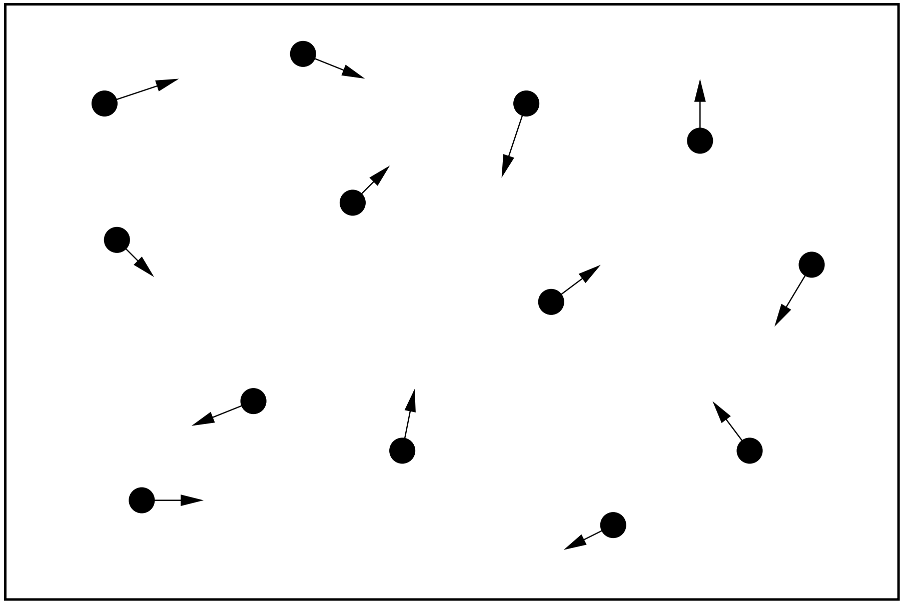
So what we found is actually

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}$$

in theories with gravity duals (no c!)
Is it large or small?
Compare to weakly coupled systems

Viscosity of dilute gases

Already Boltzmann understood the microscopic origin of viscosity. Consider a dilute gas:



Using physical arguments, Boltzmann deduced

$$\eta \sim \rho v \ell = \text{mass density} \times \text{velocity} \times \text{mean free path}$$

from which he explained the independence of viscosity on density, at fixed temperature.

Weakly coupled relativistic systems

First-principle calculation of viscosity in weakly coupled scalar field theories:
Sanyong Jeon *"Hydrodynamic transport coefficients in relativistic scalar field theory"*
Ph.D. thesis, University of Washington 1994, 162 pages

- Need infinite resummation of Feynman diagrams
- Equivalent to a kinetic Boltzmann equation
- Parametrically

$$\eta \sim \text{Energy density} \times \text{mean free time.}$$

η/s and the uncertainty principle

$$\eta \sim \rho v \ell, \qquad s \sim n = \frac{\rho}{m}$$

$$\Longrightarrow \frac{\eta}{s} \sim m v \ell = \hbar \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

But for a quasiparticle, the de mean free part cannot be smaller than the de Broglie wavelengh. So we conclude that

$$\frac{\eta}{s} > \text{some constant} \times \hbar$$

Stronger coupling $\rightarrow$ shorter mean free path $\rightarrow$ smaller viscosity.
Theories with gravity duals have, in some sense, infinite coupling.
Therefore it is natural that they would have very small, possibly smallest, η/s ratio.
So the conjecture is

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi}$$

Condensed matter analog

Mott's minimal metallic conductivity

Consider a two dimensional electron system, density n.
Conductivity:

$$\sigma \sim e^2 \frac{n}{m} \tau$$

mean free time

However:

$$\frac{n}{m} \sim \frac{\epsilon_F}{\hbar^2}, \qquad \epsilon_F = \text{Fermi energy}$$

Therefore

$$\sigma \sim \frac{e^2}{\hbar^2} \epsilon_F \tau \gtrsim \frac{e^2}{\hbar}$$

since $\epsilon_F \tau \gtrsim \hbar$.

An important remark

Viscosity *diverges* when interaction is turned off

Reason: large mean free path $\to$ easy transport of momentum.

Apparent paradox: when interaction is turned off, there is no dissipation. Why is the viscosity infinite?

Answer: the following two limits do not commute:

- The limit of infinitely weak interaction
- The hydrodynamic limit of microscopic distances

To find the viscosity as interaction $\to 0$, one needs to do experiments at larger and larger distances

Another remark

To have $\eta/s \sim \hbar$, strong coupling is necessary
but not sufficient: the system has to be also *quantum*
Strongly coupled classical plasma will have $\eta/s \gg \hbar$
In other words, three length scales have to be of the same order for $\eta/s \sim \hbar$:

(i) Mean free path

(ii) Interparticle distance

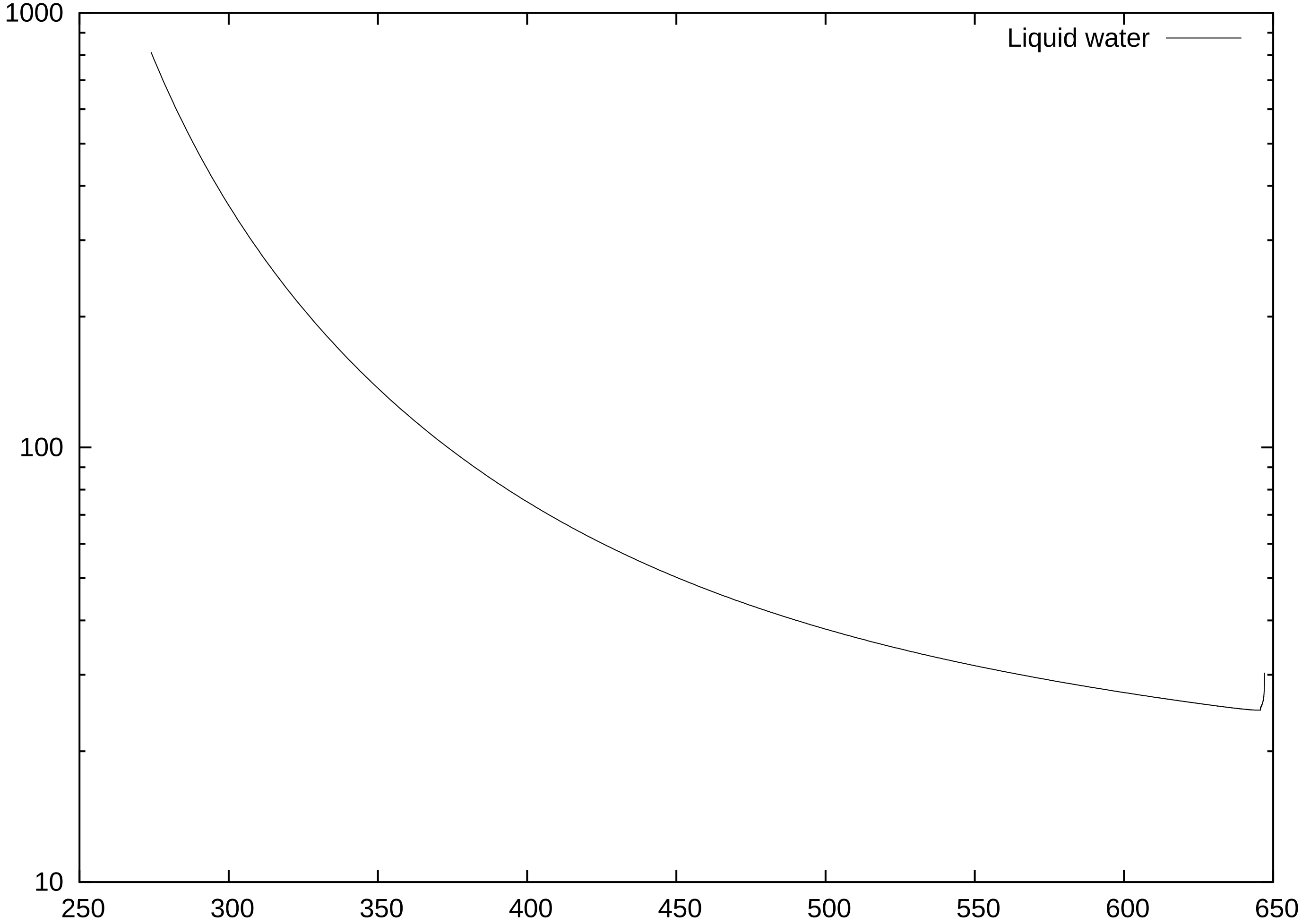
(iii) de Broglie wavelength of particles.

In strongly coupled classical plasmas, (i) and (ii) are of the same order,

but (iii) is much smaller.

η/s for ordinary liquids

η/s of liquid water in unit of $\hbar/(4\pi)$, as function of temperature (K), along the saturation curve



General observation on liquids

- η/s reaches minimum near the critical point of the liquid-gas phase transition (also: Csernai, Kapusta, McLerran)
- but not exactly at the critical point: η diverges there as (correlation length)$^{x_\eta}$, $x_\eta = 0.05 - 0.07$, according to theory and experiment
- The minimal value of η/s varies from substance to substance

Substance	$\left(\frac{\eta}{s}\right)_{\rm min}$	Substance	$\left(\frac{\eta}{s}\right)_{\rm min}$	Substance	$\left(\frac{\eta}{s}\right)_{\rm min}$
H, He	8.8				
Ne	17	H_2O	25	CO	35
Ar	37	H_2S	35	CO_2	32
Kr	57	N_2	23	SO_2	39
Xe	84	O_2	28		

(η/s is measured in unit of $\hbar/(4\pi)$)

Minimum among substances is reach by the most quantum of ones: hydrogen and helium

Not surprising since one needs the de Broglie wavelength to be as close to the mean free path as possible. The minimum is still far above $\hbar/4\pi$.

Purcell: viscosity of liquids

Life at low Reynolds number

E. M. Purcell
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 12 June 1976)

Editor's note: This is a reprint (slightly edited) of a paper of the same title that appeared in the book *Physics and Our World: A Symposium in Honor of Victor F. Weisskopf*, published by the American Institute of Physics (1976). The personal tone of the original talk has been preserved in the paper, which was itself a slightly edited transcript of a tape. The figures

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But it's more mysterious than that, Viki, because if you look at the Chemical Rubber Handbook table you will find that there is almost no liquid with viscosity much lower than that of water. The viscosities have a big range *but they stop at the same place*. I don't understand that. That's what I'm leaving for him.[1]

The matter at RHIC

Although QCD does not have gravity dual, there is a good chance that matter created at RHIC does have a small η/s ratio: during most of the evolution the temperature is $\sim \Lambda_{\text{QCD}}$.

- It is important to put give theoretical bounds (even upper bound) on η/s at RHIC.

The other strongly coupled matter

- Dilute nonrelativistic Fermi gas, 2 species (e.g., spin up and down).
- Short-range interaction, fine-tune to have one bound state at threshold: resonant s-wave interaction at low energies

$$a \to \infty, \qquad r_0 \to 0$$

- No dimensionful parameter except density. Nonrelativistic conformal invariance with interesting consequences (see e.g., DTS and M Wingate 2005).
- Two phases: superfluid (low T) and normal (high T). Around $T \sim T_c$ there should be a minimum of η/s, would be interesting to know the value.
- No reliable theoretical framework (except maybe epsilon expansion around four spatial dimensions, Y. Nishida and DTS 2006), but in principle can be determined experimentally

Conclusion

- Gauge/gravity duality, a by-product of string theory, provides unexpected tools to compute the viscosity of some strongly coupled theories
- The class of theories with gravity dual description is limited, but contains very interesting theories with infinite coupling
- The calculation of the viscosity is easy: viscosity $\propto$ absorption cross section of low-energy gravitons by the black hole.
- In this class, the ratio η/s is equal to a universal number $\hbar/4\pi$, much smaller than in any other system in Nature
- The quark-gluon-plasma near the transition from hadrons to quarks is a candidate to the system with the least η/s.
- Another candidate is dilute Fermi gas at infinite scattering length and $T \sim T_c$.

Thanks to: C. Herzog, P. Kovtun, G. Policastro, A. O. Starinets