Linear and non-linear small-x evolution in pQCD

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Based on: 1501.03754
1604.07417 (w/ Matti Herranen)

RBRC saturation workshop, Brookhaven, April. 27th 2017

Motivations

I'll be discussing higher-order perturbative corrections.

1. BFKL convergence is slow:

[~'98]

attributed to DGLAP physics -> resum.

[Salam;
Ball,Forte ~'00,…
Iancu,Mueller et al '14]

theoretically, this fits one data point.
$\Rightarrow$how well does it predict 3-loop?

Further Motivations

2. Multi-loops are standard in many QCD contexts

   (β: 4(5) loops; DGLAP: 3 loops; Higgs σ: 3loops,…)

   Q_s^2 in saturation physics never that big…

3. Purely theoretical:

 -partonic amplitudes in Regge limit:
  unique insight into scattering at high loops

 -generally interesting limit (pomeron$\rightarrow$graviton in AdS CFT,…)

 -new qualitative features @NNLL(non-planar pomeron loop…)

Outline

1. New duality between rapidity & soft evolution
   [cf Duff Neill's talk!]

2. Rapidity-Soft duality as a computation tool
   -What is needed at NLO
   -Why do they match even in QCD?

3. Outline of NNLO (so far, N=4 SYM)
   -test 1: integrability
   -test 2: collinear limits& DGLAP

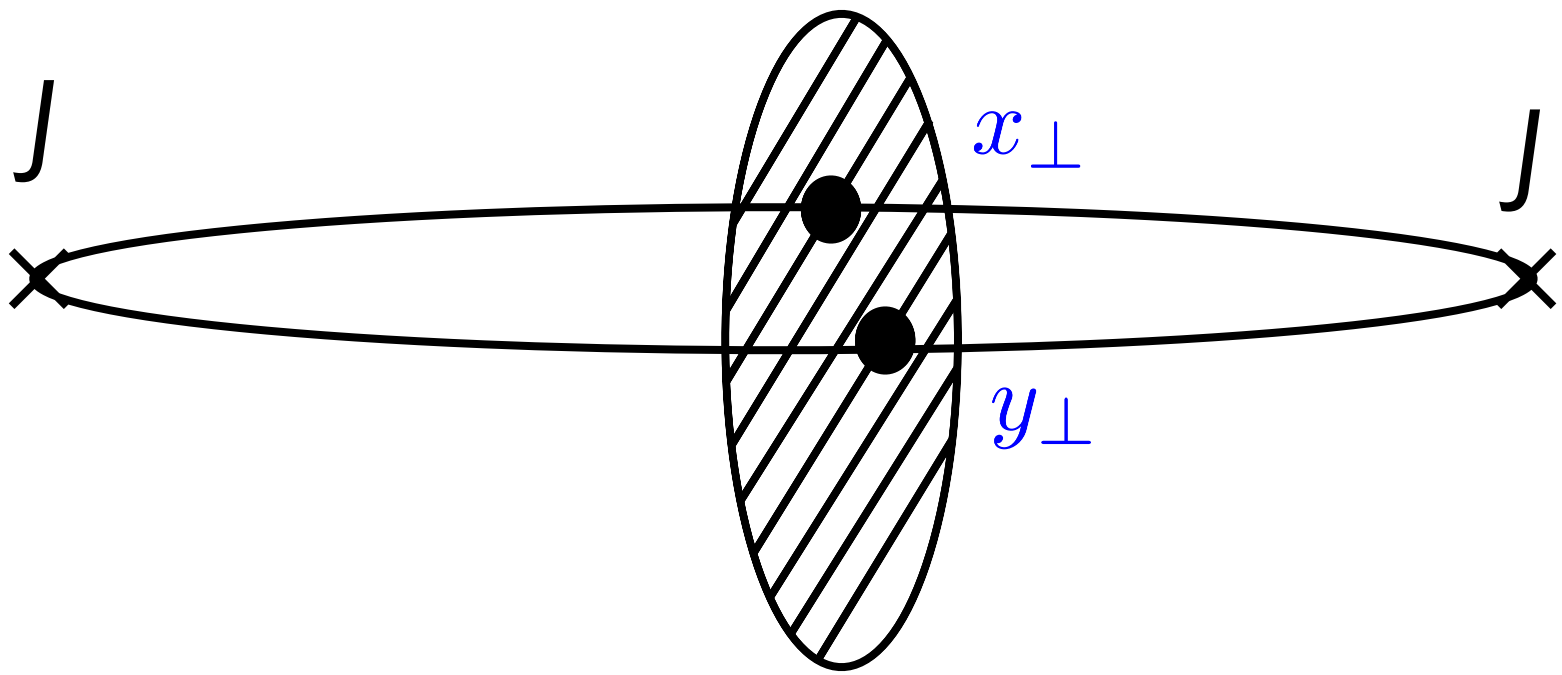
NLO BFKL&BK ('98,'07)
were hard to compute

A different formulation
would be nice

- Consider amplitude (like others in this workshop):

$$\sigma_{\mathrm{DIS}} \propto \mathrm{Im}\, A(\gamma^* p \to \gamma^* p)$$

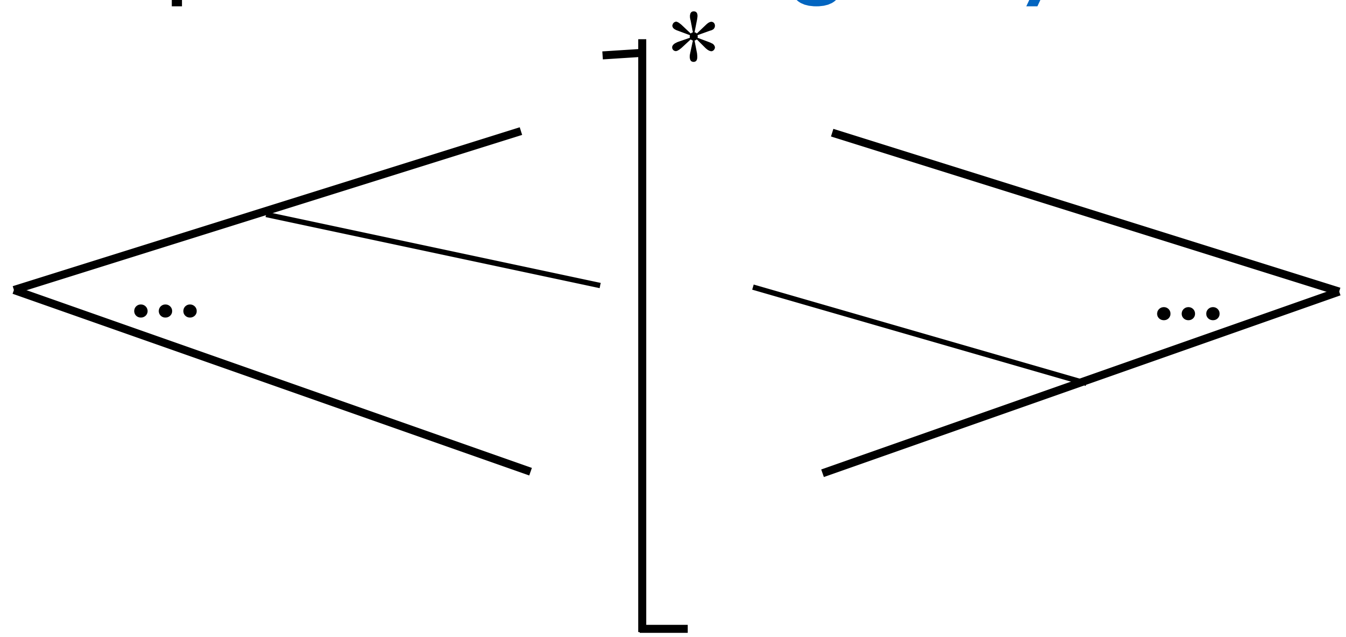
- Dipole picture: $\to \int d^2x d^2y\, \rho(x-y)\, \langle U_{xy} \rangle_{\mathrm{target}}$



[Mueller; Balitsky; Kovchegov; JIMWLK,...]

Small dipole: transparent $(U \to 1)$
Large dipole: opaque $(U \to 0)$

- General (semi-exclusive) jet observables can be phrased analogously



$$\sigma = \sum_n \int d\mathrm{Lips}_n(\{p_i\})\, u(\{\theta_i\},\{E_i\})\, |A_n|^2$$

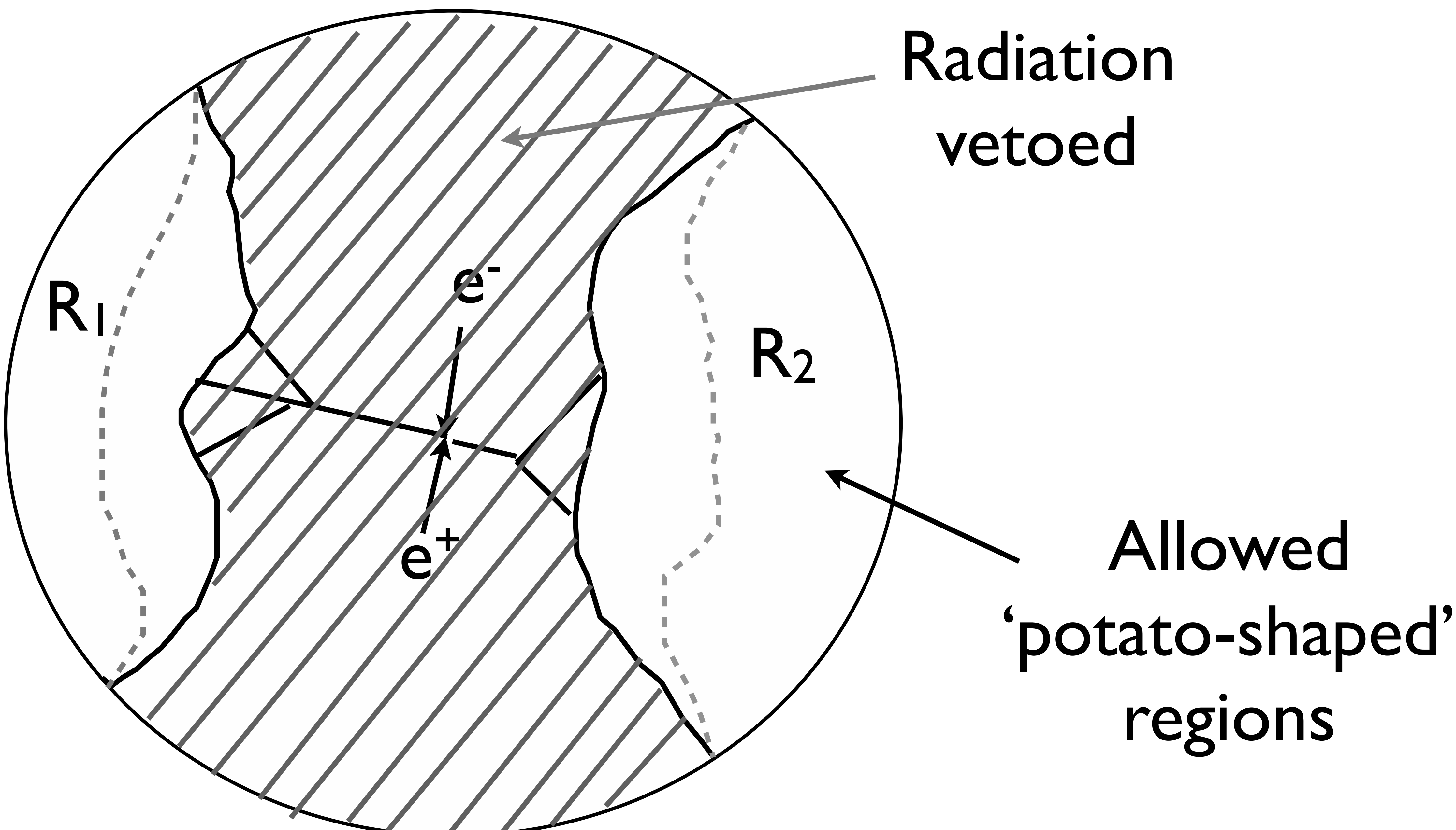
measurement ‘u’ encodes all the experimental cuts

Forward scattering ⇔ jet observables

Target	Measurement
Transparent	Allowed region
Opaque	Veto region
Rapidity Y	?

Non-global logs

Q: Cross-section for $e^+e^- \to X$, with 'X' energy smaller than E_0 outside some region R



- Archetype for some actually interesting questions ('how much energy inside a fixed cone',...)

- Suppressed by large soft* logs

$$\exp(-\#\alpha_s \log(Q/E_{\rm cut}))$$

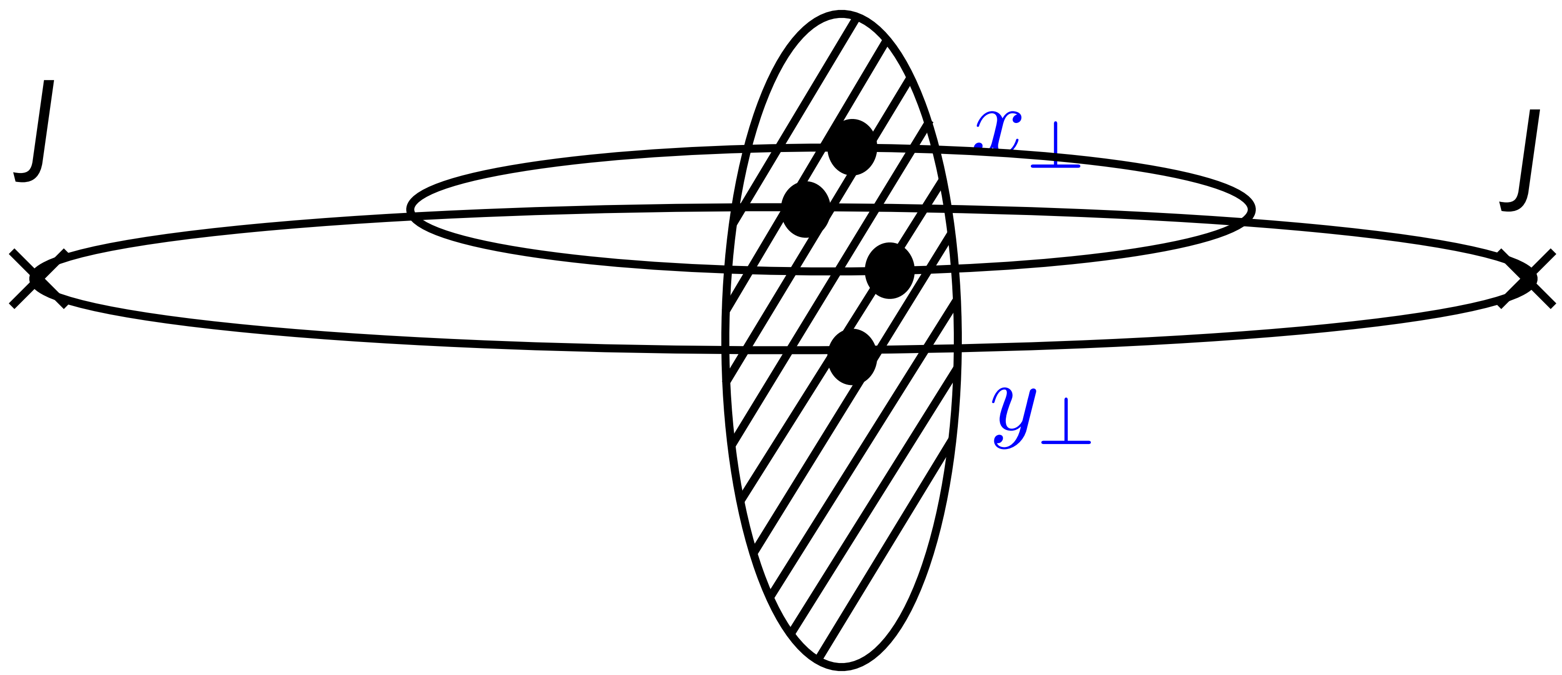
[Salam&Dasupta '01
Banfi, Salam& Dasgupta '03]

- angles not 'globally integrated'

- Difficulty: need to keep track of all radiation in allowed region! [color&angle]

*'soft'=GeV<E_{cut}<<TeV

In forward scattering we also keep all radiation

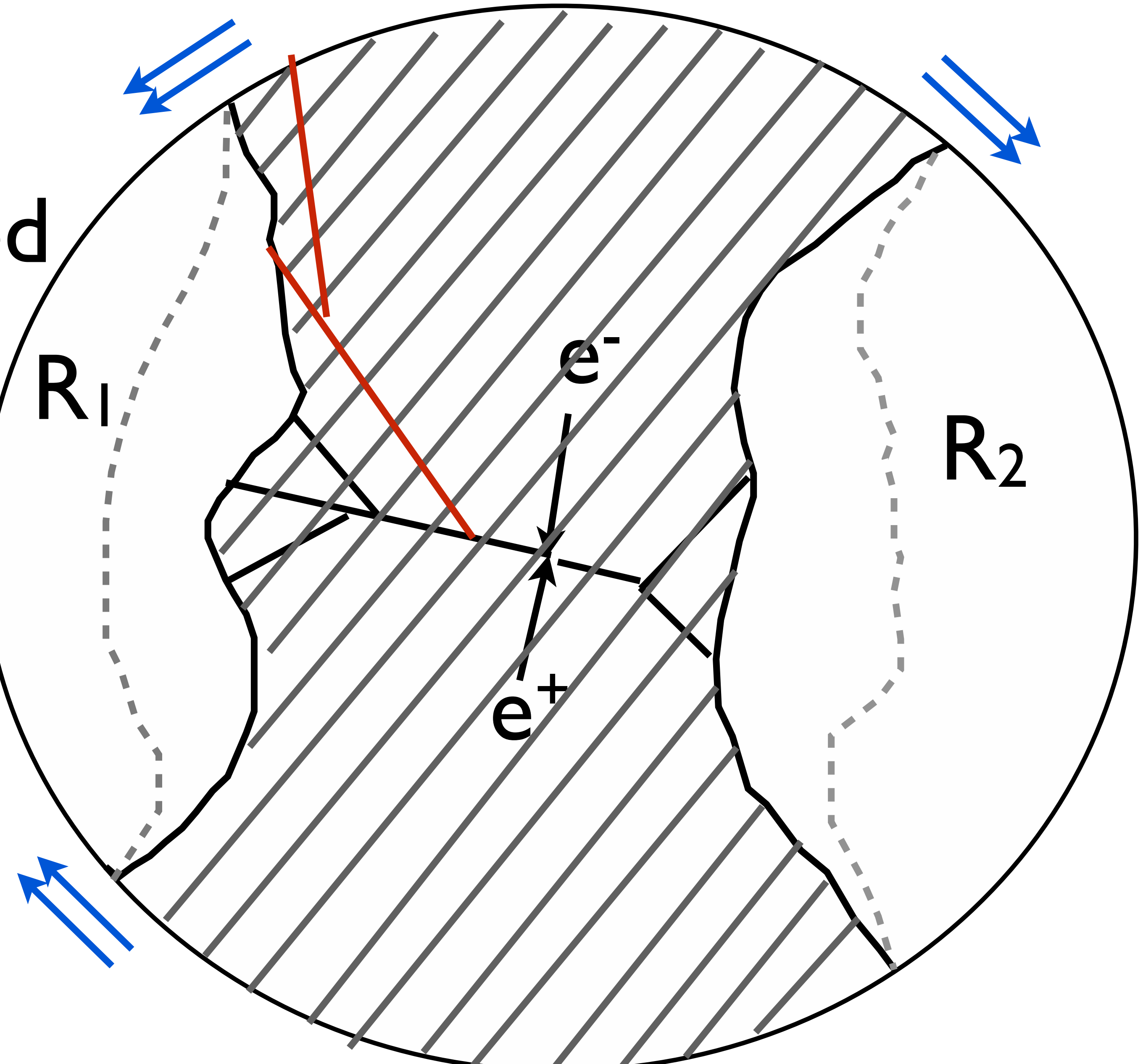


More and more dipoles
become saturated (opaque) at high rapidity

With increased energy, near-boundary jets less likely

veto region ‘opaque’:
effectively grows

Effective allowed
=‘transparent’
shrinks



Rapidity evolution (small x amplitude)	$\Leftrightarrow$	Soft evolution (small E cross-section)
Transparent		Allowed region
Opaque		Vetoed region
Rapidity Y		Soft veto
smaller dipoles saturate		effective veto region grows

- Both controlled by soft gluons
- Care not about energies, but about color
- Promote measurement functions to matrices

$$u(\theta, E) \to U^i_j(\theta)$$

(can be viewed as Wilson lines which will source softer radiation)

- Quantitative equivalence:

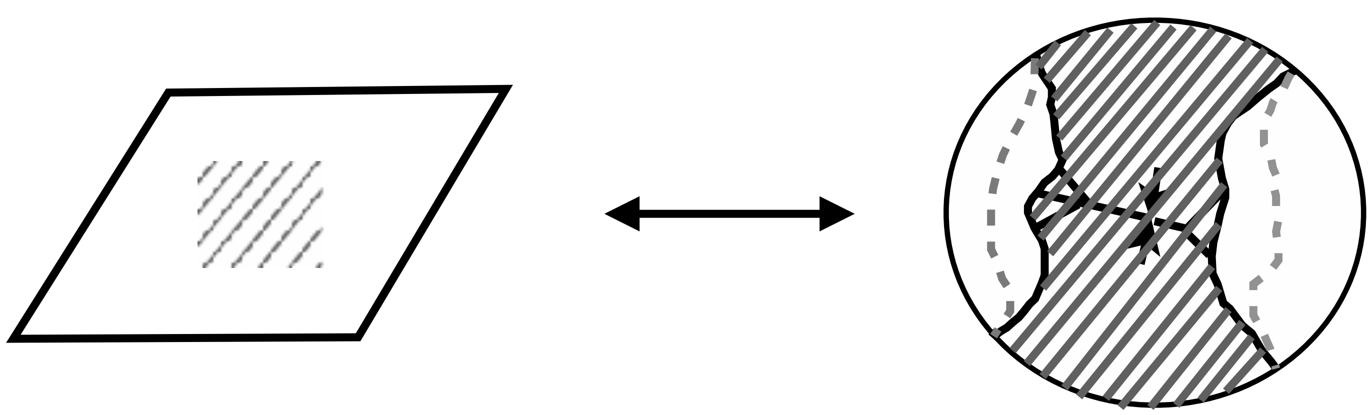
BK: $$\frac{d}{d\eta}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2 z_0}{\pi}\frac{z_{12}^2}{z_{10}z_{02}}(U_{10}U_{02} - U_{12})$$ Rapidity evolution

BMS: $$E\frac{d}{dE}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2\Omega_0}{4\pi}\frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}(U_{10}U_{02} - U_{12})$$ Soft evolution

- Conformal (stereographic) transformation:

$$\alpha_{ij} \equiv \frac{1-\cos\theta_{ij}}{2} \to z_{ij}^2 \equiv (z_i - z_j)^2, \qquad \frac{d\Omega}{4\pi} \to \frac{d^2 z}{\pi}$$

[Weigert '03; Hatta '08-...]



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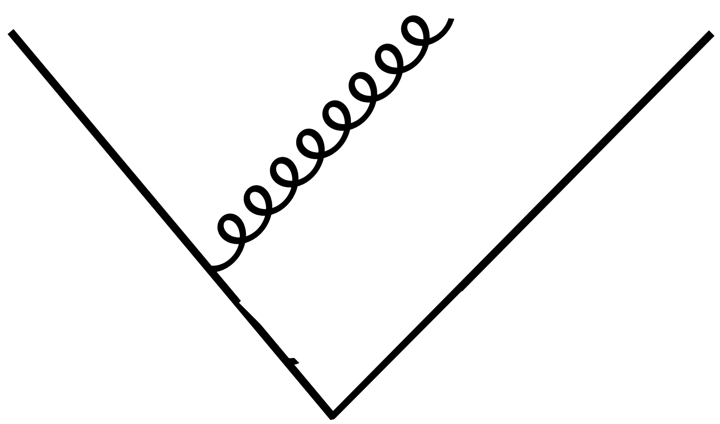
Computing non-global logs

- Soft gluon amplitude is universal:

[Weinberg]

$$\lim_{p_0 \to 0} M_{n+1} = \sum_i \frac{\epsilon \cdot p_i}{p_0 \cdot p_i} g T_i^a \times M_n$$



- For a parent dipole:

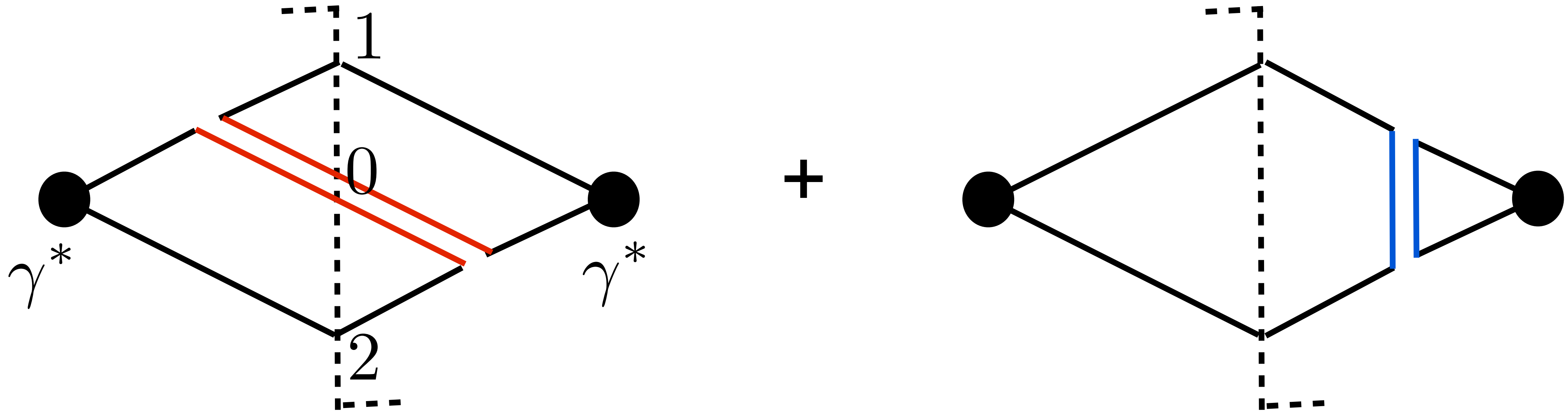
$$|M_3|^2 \simeq \frac{s_{12}}{s_{10} s_{02}} |M_2|^2$$

- Energy logs from usual IR divergent phase space:

$$\sim \log(Q/E_{\mathrm{cut}})$$

$$\int d\mathrm{Lips}(p_0)|M_3|^2 \to |M_2|^2 \int_{E_0}^{Q} \frac{dp_0}{p_0} \int \frac{d\Omega}{4\pi} \frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}$$

- Radiated gluon will induce softer radiation at later steps: dress with a Wilson line



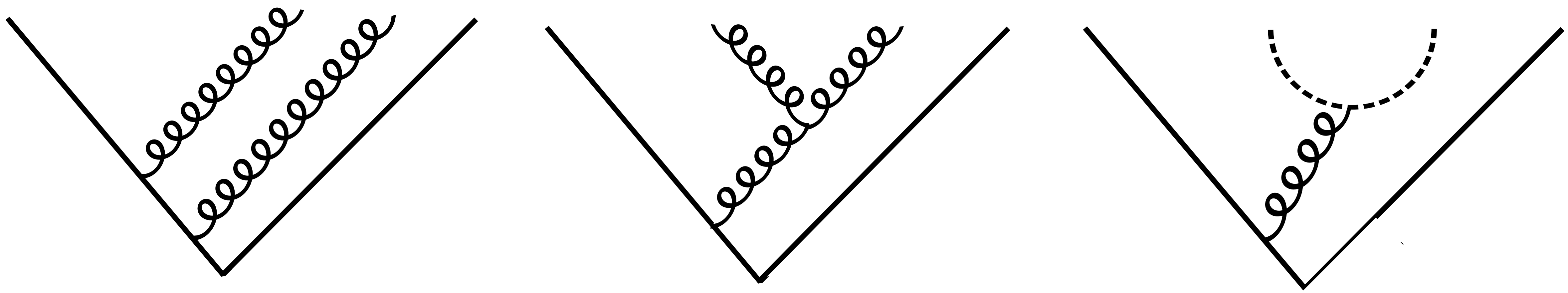
- Similar to textbook computation of IR divergences, except angular integral 'not global'!

$$E\frac{d}{dE}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2\Omega_0}{4\pi}\frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}\left(U_{10}U_{02} - U_{12}\right)$$

[BMS eq] ✔

- Real& virtual related by KLN [cancel for U=1]

NLO:



[Catani&Grazzini '99]

Square of tree-level soft current relatively simple:

$$|\mathcal{S}|^2 = \frac{s_{12}}{s_{10}s_{00'}s_{0'2}}\left[1+\frac{s_{12}s_{00'}+s_{10}s_{0'2}-s_{10'}s_{20}}{2(s_{10}+s_{10'})(s_{02}+s_{0'2})}\right]$$

N=4SYM

$$+(n_F-4)\frac{s_{12}}{s_{00'}(s_{10}+s_{10'})(s_{20}+s_{20'})}$$

$$+(2+n_s-2n_F)\frac{(s_{10}s_{20'}-s_{10'}s_{20'})^2}{2s_{00'}^2(s_{10}+s_{10'})^2(s_{20}+s_{20'})^2}$$

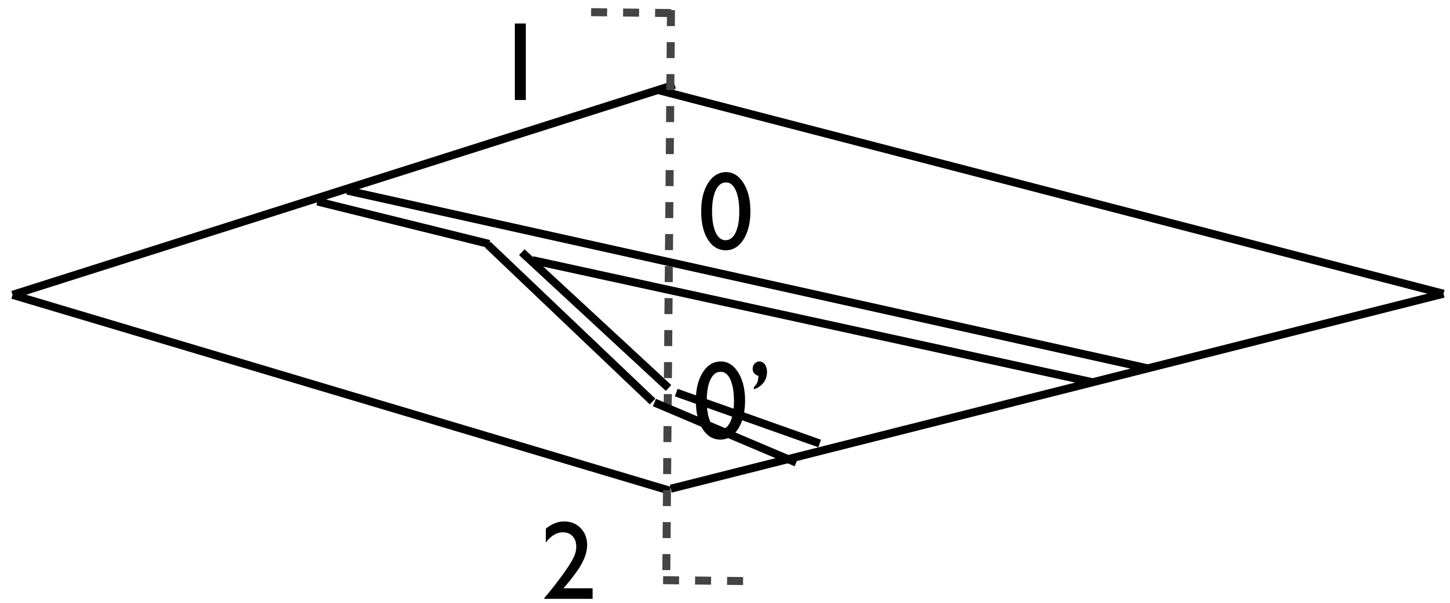
general gauge thy

[SCH, '15]

- Crucial: two soft gluons not independent

$$|\mathcal{S}|^2 = \frac{s_{12}}{s_{10}s_{00'}s_{0'2}}\left[1+\frac{s_{12}s_{00'}+s_{10}s_{0'2}-s_{10'}s_{20}}{2(s_{10}+s_{10'})(s_{02}+s_{0'2})}\right]$$

- Amplitude depends on ratio of soft gluon energies
- NLO is basically the integral over that ratio

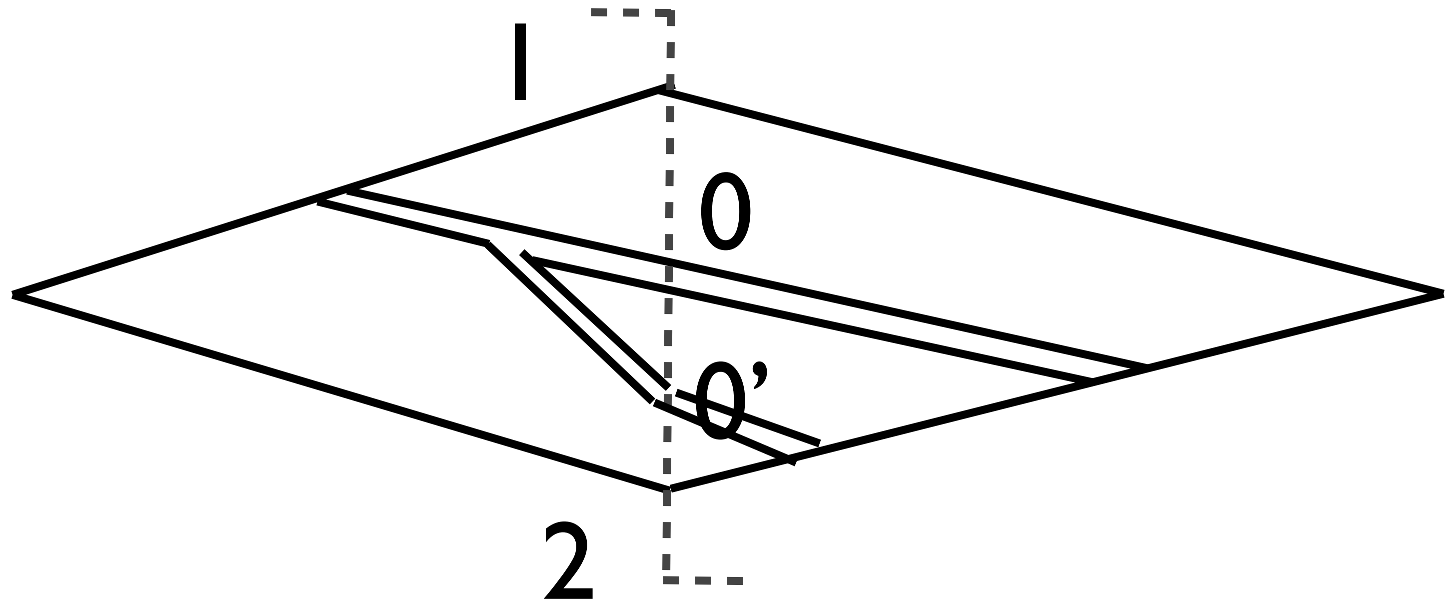


Pull out angular integrals:

$$E\frac{d}{dE}U_{12} \supset \int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K_{[1\,00'\,2]} U_{10} U_{00'} U_{0'2}$$

Integrate over relative energies:

$$K_{[1\,00'\,2]} = \int_0^\infty \tau d\tau \left[\left| \mathcal{S}(\tau\beta_0, \beta_{0'}) \right|^2 \qquad\qquad\qquad\qquad \right]$$



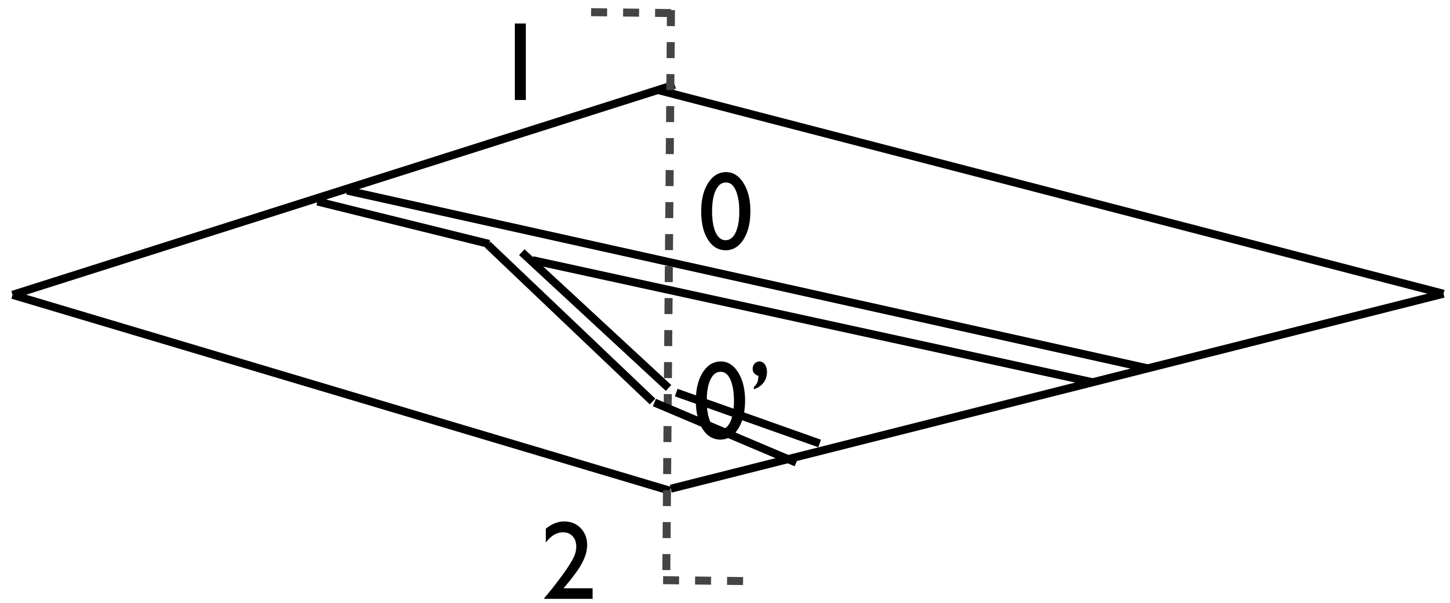
Pull out angular integrals:

$$E\frac{d}{dE}{\color{red}U_{12}} \supset \int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K_{[1\,00'\,2]}{\color{red}U_{10}U_{00'}U_{0'2}}$$

Integrate over relative energies:

$$K_{[1\,00'\,2]} = \int_0^\infty {\color{blue}\tau d\tau} \left[\left|\mathcal{S}({\color{blue}\tau}\beta_0, \beta_{0'})\right|^2 - \Big|_{{\color{blue}\tau}\to 0}\theta({\color{blue}\tau}<1) - \Big|_{{\color{blue}\tau}\to\infty}\theta({\color{blue}\tau}>1) \right]$$

Subtract iterations of LO



Pull out angular integrals:

$$E\frac{d}{dE}U_{12} \supset \int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K_{[1\,00'\,2]} U_{10}U_{00'}U_{0'2}$$

Integrate over relative energies:

$$K_{[1\,00'\,2]} = \int_0^\infty \tau d\tau \left[\begin{array}{l} \left|\mathcal{S}(\tau\beta_0,\beta_{0'})\right|^2 \\ -\Big|_{\tau\to 0}\theta(Q^2_{[1\tau 00']} < Q^2_{[10'2]}) \\ -\Big|_{\tau\to\infty}\theta(Q^2_{[00'2]} < Q^2_{1\tau 02]}) \end{array} \right]$$

Best: order w/Lorentz-invariant trans. mom $Q^2_{[i0j]} \equiv \frac{s_{i0}s_{0j}}{s_{ij}}$

- That's basically it! NLO (planar) evolution:

$$K^{(2)}U_{12} = \int_{\beta_0,\beta_{0'}} \frac{\alpha_{12}}{\alpha_{10}\alpha_{00'}\alpha_{0'2}} K^{(2)}_{[1\,00'\,2]} \left(U_{10}U_{02}+U_{10'}U_{0'2}-2U_{10}U_{00'}U_{0'2}\right) + \gamma_K^{(2)} K^{(1)} U_{12}$$

$$K^{(2)}_{[1\,00'\,2]} = 2\log\frac{\alpha_{12}\alpha_{00'}}{\alpha_{10'}\alpha_{02}} + \left(1+\frac{\alpha_{12}\alpha_{00'}}{\alpha_{10}\alpha_{0'2}-\alpha_{10'}\alpha_{02}}\right)\log\frac{\alpha_{10}\alpha_{0'2}}{\alpha_{10'}\alpha_{02}}$$ ✔

- Precisely Balitsky&Chirilli's (N=4) result!!!

[Balistky&Chirilli '07,'08]

- Eigenvalues match 'Pomeron trajectory'

[Fadin&Lipatov(&Kotikov) '98;
Ciafaloni&Gamici '98]

- Full non-planar NLO result also available (N=4&QCD)

$$
\begin{aligned}
K^{(2)} = & \int_{i,j,k}\int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K^{(2)\ell}_{ijk;00'} i f^{abc}\left(L^a_{i;0}L^b_{j;0'}R^c_k - R^a_{i;0}R^b_{j;0'}L^c_k\right) \\
& + \int_{i,j}\int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K^{(2)N=4,\ell}_{ij;00'}\left(f^{abc}f^{a'b'c'}U_0^{bb'}U_{0'}^{cc'} - \frac{C_A}{2}(U_0^{aa'}+U_{0'}^{aa'})\right)(L_i^a R_j^{a'} + R_i^{a'} L_j^a) \\
& + \int_{i,j}\int \frac{d^2\Omega_0}{4\pi}\frac{\alpha_{ij}}{\alpha_{0i}\alpha_{0j}}\gamma_K^{(2)}\left(R^a_{i;0}L^a_j + L^a_{i;0}R^a_j\right) + K^{(2)N\neq 4}. \qquad (3.32)
\end{aligned}
$$

known

[SCH '15]

$$L^a_{i;0} \equiv (L_i^{a'}U_0^{a'a} - R_i^a)$$

$$
\begin{aligned}
\alpha_{0i}\alpha_{0'j}K^{(2)\ell}_{ijk;00'} = & \frac{\alpha_{ij}}{\alpha_{00'}}\log\frac{\alpha_{0'i}\alpha_{0'j}\alpha_{0k}^2}{\alpha_{0i}\alpha_{0j}\alpha_{0'k}^2} + \frac{\alpha_{ik}\alpha_{jk}}{\alpha_{0k}\alpha_{0'k}}\log\frac{\alpha_{ik}\alpha_{0'j}\alpha_{0k}}{\alpha_{jk}\alpha_{0i}\alpha_{0'k}} + \frac{\alpha_{0'i}\alpha_{jk}}{\alpha_{00'}\alpha_{0'k}}\log\frac{\alpha_{jk}\alpha_{0i}\alpha_{00'}\alpha_{0'k}}{\alpha_{0k}^2\alpha_{0'i}\alpha_{0'j}} \\
& - \frac{\alpha_{ik}\alpha_{0j}}{\alpha_{0k}\alpha_{00'}}\log\frac{\alpha_{ik}\alpha_{0'j}\alpha_{00'}\alpha_{0k}}{\alpha_{0'k}^2\alpha_{0i}\alpha_{0j}} + \frac{\alpha_{ik}\alpha_{0'j}}{\alpha_{0'k}\alpha_{00'}}\log\frac{\alpha_{ik}\alpha_{00'}}{\alpha_{0k}\alpha_{0'i}} - \frac{\alpha_{0i}\alpha_{jk}}{\alpha_{0k}\alpha_{00'}}\log\frac{\alpha_{jk}\alpha_{00'}}{\alpha_{0'k}\alpha_{0j}} \\
K^{(2)N=4,\ell}_{ij;00'} = & \frac{\alpha_{ij}}{\alpha_{0i}\alpha_{00'}\alpha_{0'j}}\left(2\log\frac{\alpha_{ij}\alpha_{00'}}{\alpha_{0'i}\alpha_{0j}} + \left[1 + \frac{\alpha_{ij}\alpha_{00'}}{\alpha_{0i}\alpha_{0'j} - \alpha_{0'i}\alpha_{0j}}\right]\log\frac{\alpha_{0i}\alpha_{0'j}}{\alpha_{0'i}\alpha_{0j}}\right). \qquad (3.33)
\end{aligned}
$$

Precisely the same as NLO B-JIMWLK result ✔

[Kovner,Mulian&Lublinski '14,
Balitsky&Chirilli '14]

(cf's Lublinski's talk)

Upshots:

- Use building blocks that are standard within the pQCD/amplitudes community: soft currents
- (Already known to two-loops)
- All steps Lorentz-invariant
  (=SL2(C) conformal symmetry of transverse plane)
- No Fourier transform step: $\theta \leftrightarrow x_\perp$
- Agreement is both:
  -check on duality ✔
  -check on recent NLO results ✔

Wait. They look different!

$$
\begin{aligned}
H^{NLO\ JIMWLK} &= \int_{x,y} K_{2,0}(x,y)\,[J_L^a(x)\,J_L^a(y) \,+\, J_R^a(x)\,J_R^a(y)] \\
&- 2\int_{x,y,z} K_{2,1}(x,y,z)\,J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y) \\
&+ \int_{x,y,z,z'} K_{2,2}(x,y;z,z')\left[f^{abc}\,f^{def}\,J_L^a(x)\,S_A^{be}(z)\,S_A^{cf}(z')\,J_R^d(y) \,-\, N_c\,J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right] \\
&+ \int_{w,x,y,z,z'} K_{3,2}(w;x,y;z,z')f^{acb}\left[J_L^d(x)\,J_L^e(y)\,S_A^{dc}(z)\,S_A^{eb}(z')\,J_R^a(w)\right. \\
&\qquad\qquad \left. -\, J_L^a(w)\,S_A^{cd}(z)\,S_A^{be}(z')\,J_R^d(x)\,J_R^e(y)\right] \\
&+ \int_{w,x,y,z} K_{3,1}(w;x,y;z)\,f^{bde}\left[J_L^d(x)\,J_L^e(y)\,S_A^{ba}(z)\,J_R^a(w) \,-\, J_L^a(w)\,S_A^{ab}(z)\,J_R^d(x)\,J_R^e(y)\right] \\
&+ \int_{w,x,y} K_{3,0}(w,x,y)f^{bde}\left[J_L^d(x)\,J_L^e(y)\,J_L^b(w) \,-\, J_R^d(x)\,J_R^e(y)\,J_R^b(w)\right]. \qquad (2.7)
\end{aligned}
$$

[Kovner, Lublinsky & Mulian '14]

- there are relations between real&virtual

$$
K_{3,0}(w,x,y) \;=\; -\,\frac{1}{3}\left[\int_{z,z'} K_{3,2}(w,x,y;z,z') \;+\; \int_z K_{3,1}(w,x,y;z)\right]
$$

- Upshot: grouping in previous slide: all convergent

Wait. QCD is not conformal!

- QCD non-global logs in the same way
- Regge and Soft kernels don't quite agree:

$$K_{Regge} - K_{Soft} = (11C_A - 4n_F T_F - n_S T_S) \int \left(\frac{z_{ij}^2}{z_{0i}^2 z_{0j}^2} \log(\mu^2 z_{ij}^2) + \frac{z_{0j}^2 - z_{0i}^2}{z_{0i}^2 z_{0j}^2} \log \frac{z_{0i}^2}{z_{0j}^2} \right)$$

- diff prop to β = conformal breaking, as expected!

$\Rightarrow$ difference computable from matter loops!

Rapidity vs Soft divergences

- Work in d=4-2ε dimensions:

K_{Soft} does not depend on ε

$K_{Regge}(\epsilon)$ does

- In the conformal dimension, they are equal!

$$K_{Regge}(2\epsilon = -\beta(\alpha_s)) = K_{soft}$$

- Given the ε-dependence at lower loops, they are equivalent to each other!!!

[Vladimirov '16]

A slide from Ian Balitsky's talk (@Edinburgh):

NLO evolution of composite "conformal" dipoles in QCD

I. B. and G. Chirilli

$$
\begin{aligned}
a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z_3 \Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a \\
& \times \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1 + \frac{\alpha_s N_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3}\Big)\Big] \\
& + \frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2 + \frac{z_{23}{}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{23}{}^2 z_{23}^2 - z_{24}^2 z_{13}^2)}\ln\frac{z_{23}{}^2 z_{23}^2}{z_{24}^2 z_{13}^2}\Big] \\
& \times [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4 \to z_3)] \\
& + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\Big[2\ln\frac{z_{12}^2 z_{34}^2}{z_{23}{}^2 z_{23}^2} + \Big(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}{}^2 z_{23}^2}\Big)\ln\frac{z_{13}^2 z_{24}^2}{z_{23}{}^2 z_{23}^2}\Big] \\
& \times [\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4 \to z_3)]\Big\}
\end{aligned}
$$

=O(eps)term in LO BK

'conformal QCD' bit

N=4 bit

$$b = \frac{11}{3}N_c - \frac{2}{3}n_f$$

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel.

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NNLO

[Herranen+SCH, '16]

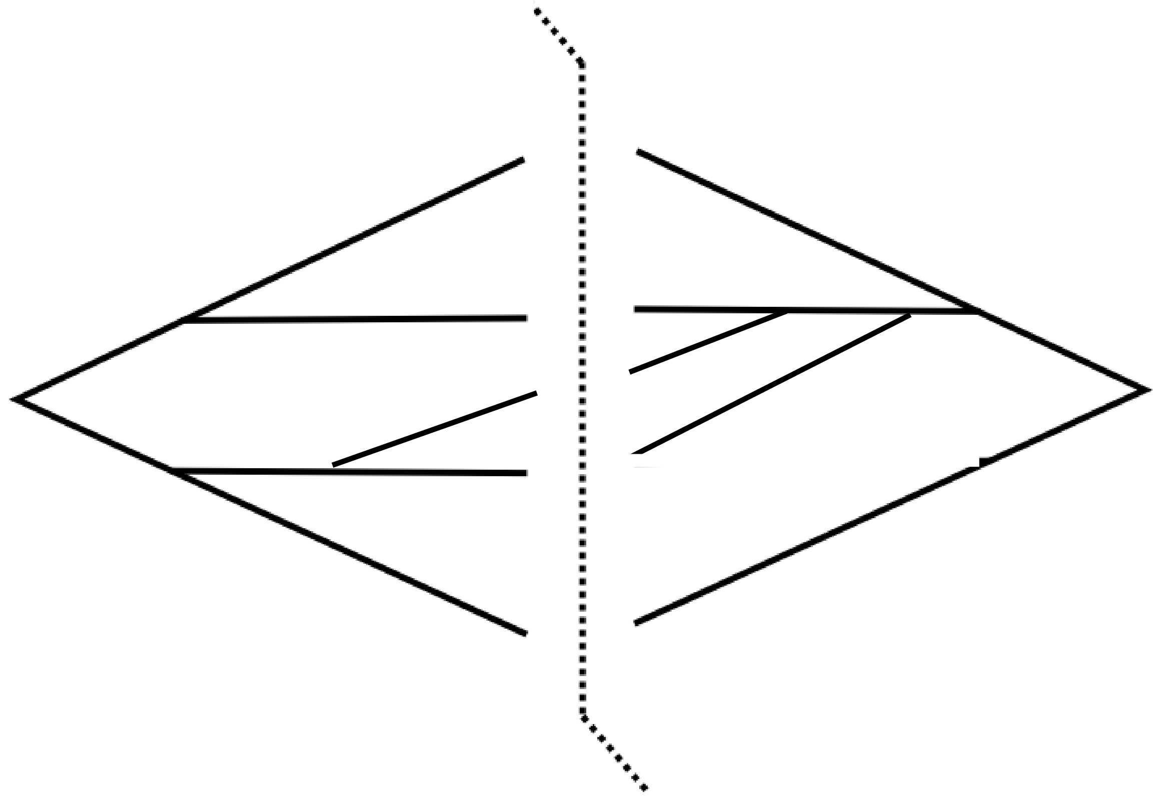
- Triple soft current at tree-level
  $\Rightarrow$ extract from known 4-particle integrand ✓
- Double soft current at one-loop
  $\Rightarrow$ extract from known one-loop 6-point ✓

[~'94]

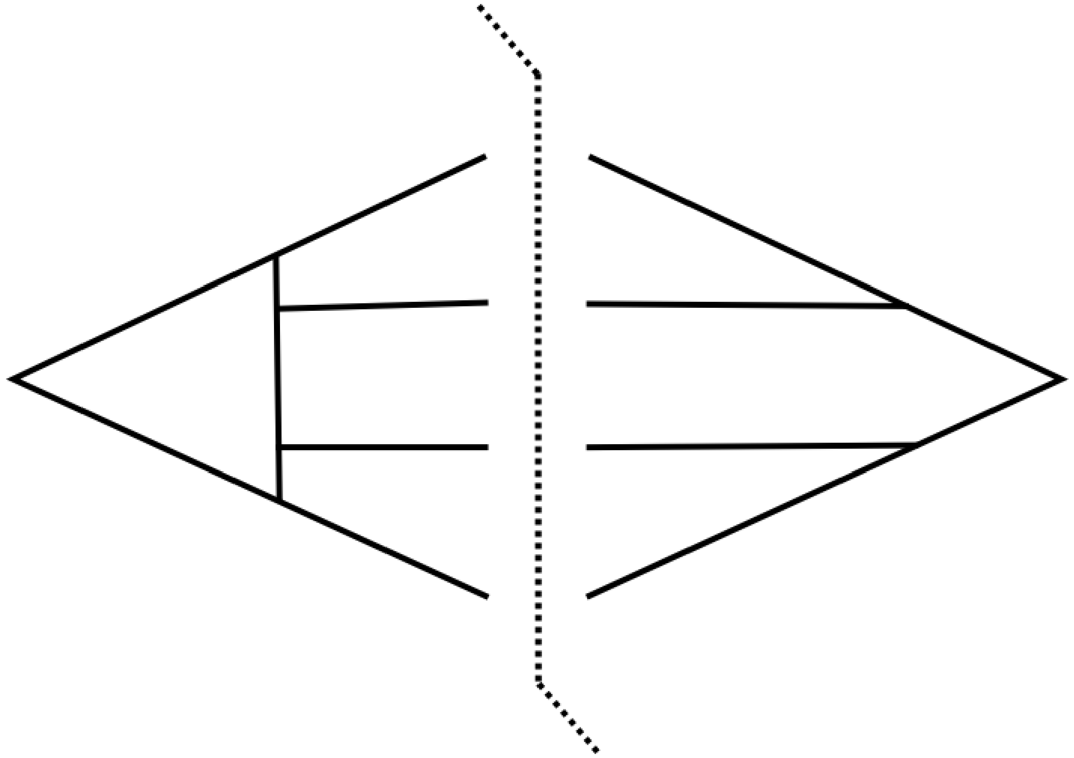
- Single soft current at two-loops
  $\Rightarrow$ not needed: contribution really just $\gamma_K^{(3)}$ ✓
- Fully virtual IR divergences at three-loops
  $\Rightarrow$ not needed: KLN fixes it from rest ✓

- Sample graphs we computed/borrowed:

triple soft emission (squared) at tree-level



double soft emission: tree/one-loop interference



- Recursive subtraction of subdivergences:

$$
\begin{aligned}
F^{\rm sub}_{[1\,0\,2]} &\equiv F_{[1\,0\,2]} = 1, && (4.20a)\\
F^{\rm sub}_{[1\,00'\,2]} &\equiv F_{[1\,00'\,2]} - [1\,0\,0'][1\,0'\,2] - [0\,0'\,2][1\,0\,2], && (4.20b)\\
F^{\rm sub}_{[1\,00'0''\,2]} &\equiv F_{[1\,00'0''\,2]} - [1\,0\,0'][1\,0'0''\,2] - [0\,0'\,0''][1\,00''\,2] - [0'\,0''\,2][1\,00'\,2]\\
&\quad -[1\,00'\,0''][1\,0''\,2] - [0\,0'0''\,2][1\,0\,2]\\
&\quad -[1\,0\,0'][1\,0'\,0''][1\,0''\,2] - [0'\,0''\,2][0\,0'\,2][1\,0\,2] - [0\,0'\,0''][1\,0\,0''][1\,0''\,2]\\
&\quad -[0\,0'\,0''][0\,0''\,2][1\,0\,2] - [1\,0\,0'][0'\,0''\,2][1\,0'\,2] - [0'\,0''\,2][1\,0\,0'][1\,0'\,2]. && (4.20c)
\end{aligned}
$$

energy step functions

- Cleanly removes iterations of lower-loop evolution

✓

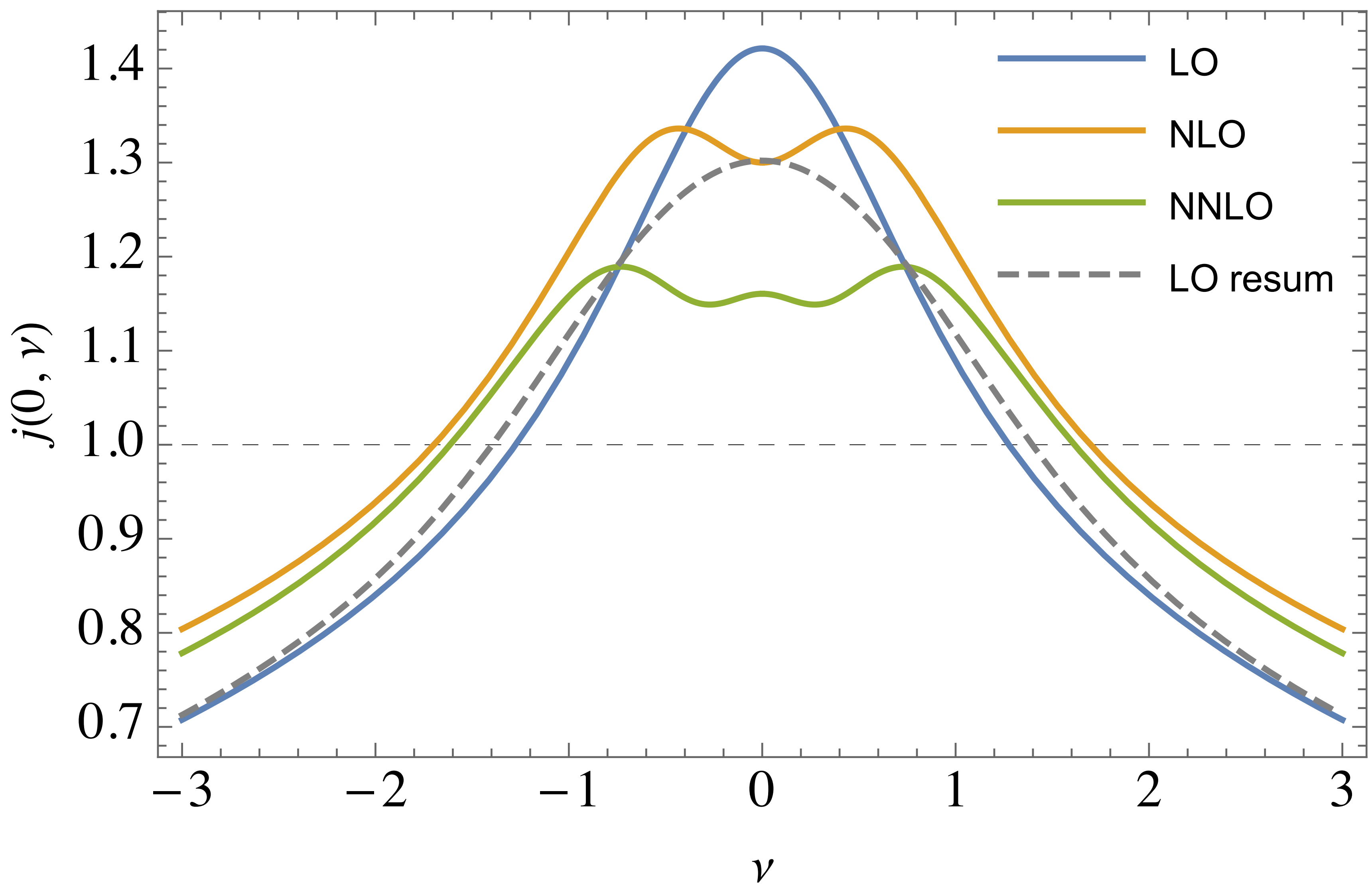
We computed only finite absolutely convergent integrals

result:

$$K^{(3)}_{[1\,00'\,2]} = \left(1 - \frac{u}{1-v}\right) \log v \left[\log u \log \frac{v}{u} - \frac{1}{3}\log^2 v - 4\zeta_2\right] + 2(1+v-u)\left(\zeta_2 \log \frac{u}{v} - 2\zeta_3\right)$$
$$+ \left(\frac{2u}{1-v} + v - u - 1\right)\left[4\mathrm{Li}_3\left(1 - \frac{1}{v}\right) + 2\mathrm{Li}_2\left(1 - \frac{1}{v}\right)\log \frac{v}{u}\right] - \frac{5}{6}\log^3 u$$
$$+4\big(\mathrm{Li}_3(x) + \mathrm{Li}_3(\bar{x}) - 2\zeta_3\big) - 2\big(\mathrm{Li}_2(x) + \mathrm{Li}_2(\bar{x}) + 2\zeta_2\big)\log u\,. \tag{4.35}$$

$$K^{(3)}_{[1\,00'0''\,2]} = \left(1 - \frac{u_3}{1 - v_1 v_2}\right)\left[\begin{array}{l} 2\mathrm{Li}_2\left(1 - \frac{1}{v_1 v_2}\right) - 2\mathrm{Li}_2\left(1 - \frac{1}{v_1}\right) - 2\mathrm{Li}_2\left(1 - \frac{1}{v_2}\right) \\ + \log v_1 \log v_2 + \log(v_1 v_2)\big(\log(u_1 u_2) - \frac{3}{2}\log u_3\big) \end{array}\right]$$
$$+ (u_1 u_2 - u_1 v_2 - u_2 v_1 + v_1 + v_2 - u_1 - u_2 + u_3)\left[\mathrm{Li}_2\left(1 - \frac{1}{v_1 v_2}\right) - \zeta_2\right]$$
$$+3\log u_1 \log u_2 - \tfrac{3}{2}\log^2 u_3 + (1+P)(f + f_1), \tag{4.23}$$

Fast to evaluate, attached in computer-friendly format to arXiv submission.



- Pomeron trajectory = linearized eigenvalue

$$U_{ij} = 1 - \frac{1}{N_c}\mathcal{U}_{ij}$$

for eigenfunction: $\mathcal{U}_{m,\nu} = |z_i - z_j|^{i\nu} e^{im\arg(z_i - z_j)}$

$$\frac{d}{d\eta}\mathcal{U}_{m,\nu} = [j(m,\nu) - 1]\mathcal{U}_{m,\nu} \qquad (\Delta = 2 + i\nu)$$

[see Brower,Polchinski,Strassler&Tan]

Tests

- Collinear limit $\nu \to \pm i$ controlled by small-x limit of DGLAP

[Jaroscewicz '83; Ball, Falgari, Forte, Marzani… 07]

$$\omega^{(3)} \to +g^6\Big(\frac{1024}{\gamma^5} - \frac{512}{\gamma^3}\zeta_2 + \frac{576}{\gamma^2}\zeta_3 - \frac{464}{\gamma}\zeta_4 + 840\zeta_5 + 64\zeta_2\zeta_3 + \gamma\Big(-40{\zeta_3}^2 - 373\zeta_6\Big) + \gamma^2\left(-8\zeta_2\zeta_5 - 86\zeta_3\zeta_4 + \frac{1001}{4}\zeta_7\right)\Big). \quad (21)$$

✔

[Velizhanin '15]

- Analytic expression for m=0 conjectured using Integrability of planar N=4

$$\frac{F^{(3)}_{0,\nu}}{32} = -S_5 + 2S_{-4,1} - S_{-3,2} + 2S_{-2,3} - S_{2,-3} - 2S_{3,-2} + 4S_{-3,1,1} + 4S_{1,-3,1} + 2S_{1,-2,2} + 2S_{1,2,-2} + 2S_{2,1,-2} - 8S_{1,-2,1,1} + \zeta_2\big(S_1S_2 - 3S_{-3} + 2S_{-2,1} - 4S_{1,-2}\big) - \tfrac{49}{2}\zeta_4 S_1 + 7\zeta_3\big(2S_{1,-1} + 2(S_1 - S_{-1})\log 2 - S_{-2} - \log^2 2\big) + (8\zeta_{-3,1} - 17\zeta_4)\big(S_{-1} - S_1 + \log 2\big) - \tfrac{1}{2}\zeta_3 S_2 + 4\zeta_5 - 6\zeta_2\zeta_3 + 8\zeta_{-3,1,1}\,. \quad (C.3)$$

✔

[new result for m>0]

[Gromov, Levkovich-Maslyuk & Sizov, '15]

[more on DGLAP vs BFKL: use *dimensions* instead of γ]

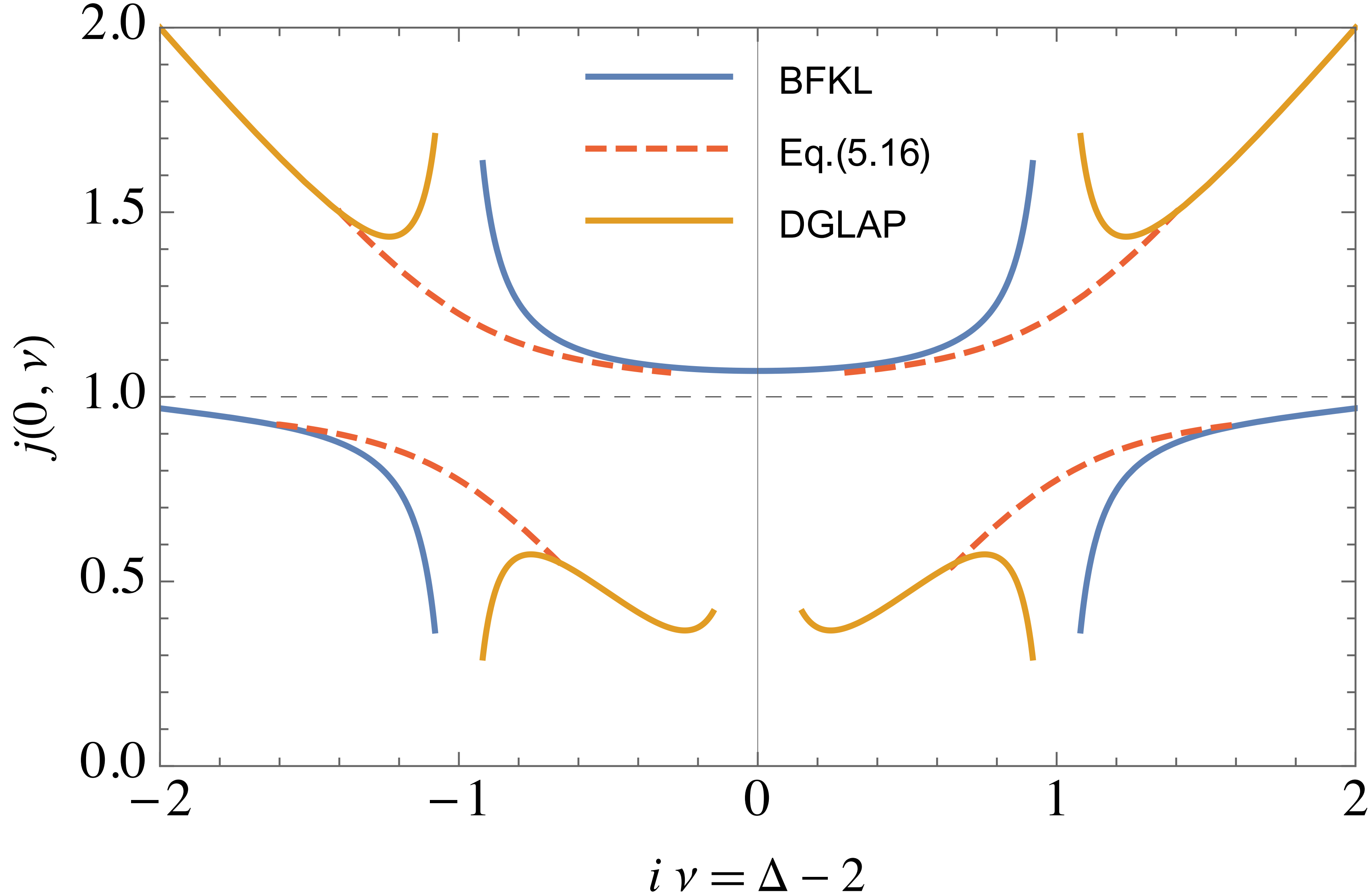


Figure 6. Level repulsion between the Pomeron and DGLAP trajectories for $m = 0$ as a function of scaling dimension, illustrating the $\nu = \pm i$ singularities. (LO expressions plotted with $\lambda = g^2_{\rm YM} N_c = 1$.)

$$j \approx 1 + \frac{\Delta - 3 \pm \sqrt{(\Delta-3)^2 + 32g^2}}{2}, \qquad \Delta = 2 + i\nu. \tag{5.16}$$

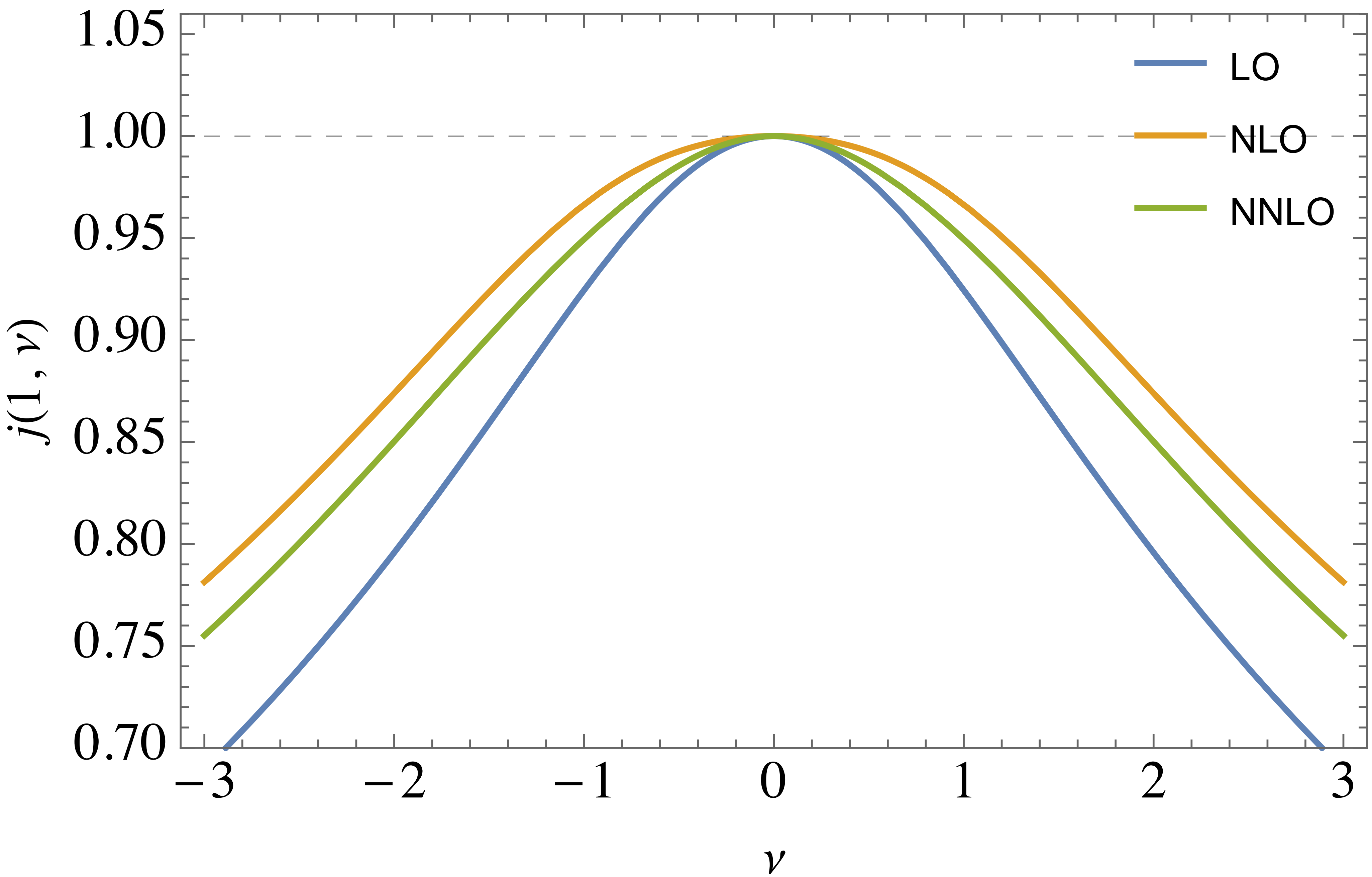
[for polarized PDFs: level crossing is at ν=0] Bartels, Ermolaev&Ryskin '96]

[cf Sievert & Kovchegov's talks]

Conclusions

- Established equivalence between two evolutions:
  Rapidity $\Leftrightarrow$ Soft (in pQCD)
- NNLL Evolution now known in planar N=4 SYM:
  -linear eigenvalue for all m=0,1,2,3,…
  -include nonlinear interactions
- QCD now in sight
- Study convergence& resummations?
- Extend duality to impact factors?

m=1 (leading Odderon trajectory)



note: Odderon intercept=1 to all orders in λ.
Agrees with strong coupling!

[Tan et al '14]

On the Odderon intercept

- m=1,ν=0 is a very special wavefunction:

$$\mathcal{U}_{12} = 1 - \frac{1}{N_c}(z_1 - z_2)$$

- Strings of dipoles in planar limit telescope:

$$\mathcal{U}_{10}\mathcal{U}_{02} = 1 - \frac{1}{N_c}((z_1 - \cancel{z_0}) + (\cancel{z_0} - z_2)) + O(1/N_c^2)$$
$$= 1 - \frac{1}{N_c}(z_1 - z_2) = \mathcal{U}_{12}$$

$$\mathcal{U}_{10}\mathcal{U}_{00'2}\mathcal{U}_{0'2} = \mathcal{U}_{12}$$

...

- Cancel in evolution. Thm: Odderon intercept vanishes to all order in λ in planar limit

matter loop contributions to NGLs:

$$
\begin{aligned}
K^{(2)N\neq 4} &= \int_{i,j}\int \frac{d\Omega_0}{4\pi}\frac{d\Omega_{0'}}{4\pi}\frac{1}{\alpha_{00'}}\left[\frac{\alpha_{ij}\log\frac{\alpha_{0i}\alpha_{0'j}}{\alpha_{0'i}\alpha_{0j}}}{\alpha_{0i}\alpha_{0'j}-\alpha_{0'i}\alpha_{0j}}\right](L_i^a R_j^{a'}+R_i^{a'}L_j^a) \\
&\quad\times\Big\{2n_F\mathrm{Tr}_R\big[T^a U_0 T^{a'} U_{0'}^\dagger\big]-4f^{abc}f^{a'b'c'}U_0^{bb'}U_{0'}^{cc'}-(n_F T_R-2C_A)(U_0^{aa'}+U_{0'}^{aa'})\Big\} \\
&+\int_{i,j}\int \frac{d\Omega_0}{4\pi}\frac{d\Omega_{0'}}{4\pi}\frac{1}{2\alpha_{00'}^2}\left[\frac{\alpha_{0i}\alpha_{0'j}+\alpha_{0'i}\alpha_{0j}}{\alpha_{0i}\alpha_{0'j}-\alpha_{0'i}\alpha_{0j}}\log\frac{\alpha_{0i}\alpha_{0'j}}{\alpha_{0'i}\alpha_{0j}}-2\right](L_i^a R_j^{a'}+R_i^{a'}L_j^a) \\
&\quad\times\left\{\begin{matrix} 2(n_S-2n_F)\mathrm{Tr}_R\big[T^a U_0 T^{a'} U_{0'}^\dagger\big]+2f^{abc}f^{a'b'c'}U_0^{bb'}U_{0'}^{cc'} \\ -((n_S-2n_F)T_R+C_A)(U_0^{aa'}+U_{0'}^{aa'}) \end{matrix}\right\} \\
&+\int_{i,j}2\pi i b_0\log(\alpha_{ij})\big(L_i^a L_j^a-R_i^a R_j^a\big). \qquad (3.34)
\end{aligned}
$$