Quark Helicity Evolution at Small $x$

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RBRC Workshop: Saturation: Recent Developments, New Ideas, and Measurements

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• Quark polarization not well constrained below $x \leq 10^{-2}$

adapted from Aschenauer et al., Phys. Rev. D92 (2015) no.9 094030
Small-x Enhancement of Quark Polarization

Quark polarization not well constrained below $x \leq 10^{-2}$

New small-x evolution can lead to significant enhancement

adapted from Aschenauer et al., Phys. Rev. D92 (2015) no.9 094030
An Appetizer:

Small-x Evolution of the Unpolarized Quark Distribution
Factorization: One-to-one correspondence between the DIS cross section and the parton distribution functions

\[
\frac{Q^2}{4\pi^2 \alpha_{EM}} \frac{d\sigma(\gamma^* p)}{dx \ dQ^2} = F_2(x, Q^2) \overset{L.O.}{=} \sum_f e_f^2 x q_f(x, Q^2)
\]
**Factorization**: One-to-one correspondence between the DIS cross section and the parton distribution functions

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\frac{Q^2}{4\pi^2\alpha_{EM}} \frac{d\sigma(\gamma^* p)}{dx \, dQ^2} = F_2(x, Q^2) \overset{L.O.}{=} \sum_f e_f^2 \, x q_f(x, Q^2)
\]

\[
\frac{d\sigma(\gamma^* p)}{dx \, dQ^2} \sim \left( \frac{1}{x} \right)^{\alpha_P-1} \sim x \, q_f(x, Q^2)
\]
The Unpolarized Dipole Amplitude

\[ q(x, Q^2) = \int \frac{d\tau^-}{2\pi} e^{\text{i} x p^+ r^-} \langle p | \bar{\psi}(0) \mathcal{U}[0, r^-] \frac{\gamma^+}{2} \psi(r^-) | p \rangle \]

- The DIS cross section / PDF is expressed in terms of a dipole scattering amplitude amplitude / cross section

\[ xq_f(x, Q^2) \stackrel{L.O.}{=} \frac{Q^2 N_c}{2\pi^2 \alpha_{EM}} \int \frac{d^2 x_{10} \, dz}{4\pi z(1-z)} \left[ |\Psi_{T,f}(x_{10}^2, z)|^2 + |\Psi_{L,f}(x_{10}^2, z)|^2 \right] \int d^2 b_{10} (1 - S_{10}(zs)) \]
The Unpolarized Dipole Amplitude

The DIS cross section / PDF is expressed in terms of a dipole scattering amplitude amplitude / cross section

$$q(x, Q^2) = \int \frac{dr^-}{2\pi} e^{i p^+ r^-} \langle p | \bar{\psi}(0) U[0, r^-] \frac{\gamma^+}{2} \psi(r^-) | p \rangle$$

$$x q_f(x, Q^2) \overset{L.O.}{=} \frac{Q^2 N_c}{2\pi^2 \alpha_{EM}} \int \frac{d^2 x_{10} \, dz}{4\pi z(1-z)} \left[ |\Psi_{T,f}(x_{10}^2, z)|^2 + |\Psi_{L,f}(x_{10}^2, z)|^2 \right] \int d^2 b_{10} (1 - S_{10}(zs))$$

$$S_{10}(zs) \equiv \left\langle \frac{1}{N_c} \text{tr}[V_{x_0} V_{x_1}^\dagger](zs) \right\rangle = 1 - \frac{1}{2} \frac{d\sigma(q_{x_0}^{unp} \bar{q}_{x_1}^{unp})}{d^2 b_{10}}(zs)$$
The Unpolarized Dipole Amplitude

\[ \gamma^* \]

\[ p \]

\[ q(x, Q^2) = \int \frac{dr^+}{2\pi} e^{i x^+ r^-} \langle p | \bar{\psi}(0) U[0, r^-] \frac{\gamma^+}{2} \psi(r^-) | p \rangle \]

- The DIS cross section / PDF is expressed in terms of a dipole scattering amplitude amplitude / cross section

\[ xq_f(x, Q^2) \overset{L.O.}{=} \frac{Q^2 N_c}{2\pi^2 \alpha_{EM}} \int \frac{d^2 x_{10} \, dz}{4\pi z (1 - z)} \left[ |\Psi_{T,f}(x_{10}^2, z)|^2 + |\Psi_{L,f}(x_{10}^2, z)|^2 \right] \int d^2 b_{10} (1 - S_{10}(z s)) \]

\[ S_{10}(z s) \equiv \left\langle \frac{1}{N_c} \text{tr} [V_{x_0}^\dagger V_{x_1}](z s) \right\rangle = 1 - \frac{1}{2} \frac{d\sigma(q_{x_0}^{unp} \bar{q}_{x_1}^{unp})}{d^2 b_{10}}(z s) \]

**Wilson lines:** \[ V_x = P \exp \left[ ig \int dz^- \hat{A}^+(0^+, z^-, x) \right] \]
Initial conditions from the quark target model:

\[ S_{10}^{(0)}(z_0) = \frac{2\alpha_s^2 C_F}{N_c} \ln^2 \frac{x_{0T}}{x_{1T}} \]
Origins of Unpolarized Evolution

- **Initial conditions from** the quark target model:

\[ S^{(0)}_{10}(zs) = \frac{2\alpha_s^2 C_F}{N_c} \ln^2 \frac{x_0 T}{x_1 T} \]

- **Soft gluon emission** spans the full rapidity interval

\[ Y \sim \ln \frac{s}{\Lambda^2} \sim \ln \frac{1}{x} \]

\[
\frac{1}{N_c} \text{tr}[V_{x_0} V_{x_1}^\dagger](zs) \sim \alpha_s \int_0^z \frac{dz'}{z'} \int d^2x_2 K(x_0, x_1, x_2) \mathcal{O} \]

\[ \ln \frac{zs}{\Lambda^2} \]
• Initial conditions from the quark target model:

\[ S_{10}^{(0)}(zs) = \frac{2\alpha_s^2 C_F}{N_c} \ln^2 \frac{x_{0T}}{x_{1T}} \]

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\[
\frac{1}{N_c} \text{tr}[V_{x_0} V_{x_1}](zs) \sim \alpha_s \int \frac{dz'}{z'} \int d^2x_2 \mathcal{K}(x_0, x_1, x_2) \mathcal{O} \]

\[ \ln \frac{zs}{\Lambda^2} \]

• Successive emissions continue to generate a logarithm of energy if they are ordered longitudinally

\[ z \gg z' \gg z'' \gg \cdots \]

• Unpolarized evolution is leading-logarithmic

\[ \alpha_s \ln \frac{1}{x} \sim 1 \]
Unpolarized Small-$x$ Evolution

\[
S_{10}(zs) = S^{(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_2} V_{x_1}^\dagger] \text{tr}[V_{x_0} V_{x_2}^\dagger] \right\rangle_{(z',s)} - S_{10}(z's) \right]
\]
Unpolarized Small-$x$ Evolution

\[ S_{10}(z s) = S_{10}^{(0)}(z s) + \left\{ \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2 x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[ \frac{1}{N_c^2} \left< \text{tr}[V_{x_2} V_{x_1}^\dagger] \text{tr}[V_{x_0} V_{x_2}^\dagger] \right>_{(z',s)} - S_{10}(z',s) \right] \right\} \]
Unpolarized Small-x Evolution

\[ S_{10}(zs) = S_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2 x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_2} V_{x_1}^{\dagger}] \text{tr}[V_{x_0} V_{x_2}^{\dagger}] \right\rangle_{(zs')} \right] - S_{10}(zs') \]
Unpolarized Small-x Evolution

\[ S_{10}(zs) = S_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_2} V_{x_1}^\dagger] \text{tr}[V_{x_0} V_{x_2}^\dagger] \right\rangle_{(z's)} \right] - S_{10}(z's) \]
Solution: The Pomeron Intercept

- Operator hierarchy closes in the large-$N_c$ limit (BK)

\[ S_{10}(zs) = S_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int^z \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} [S_{12}(z's) S_{20}(z's) - S_{10}(z's)] \]
Solution: The Pomereron Intercept

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S_{10}(zs) = S_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[ S_{12}(z's) S_{20}(z's) - S_{10}(z's) \right]
\]

- Dilute / linearized regime (BFKL):

\[
S_{10}(zs) = S_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[ S_{12}(z's) + S_{20}(z's) - S_{10}(z's) - 1 \right]
\]
Solution: The Pomeron Intercept

- Operator hierarchy closes in the large-$N_c$ limit (BK)

\[ S_{10}(zs) = S_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \frac{x_{20}^2}{x_{12}^2 x_{20}^2} \left[ S_{12}(z's) S_{20}(z's) - S_{10}(z's) \right] \]

- Dilute / linearized regime (BFKL): \( 1 - S_{10} \ll 1 \)

\[ S_{10}(zs) = S_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[ S_{12}(z's) + S_{20}(z's) - S_{10}(z's) - 1 \right] \]

- Analytic solution by Laplace/Mellin transform (intercept):

\[ x q_f(x, Q^2) \sim S_{10}(s = \frac{Q^2}{x}) \sim \left( \frac{1}{x} \right)^{\alpha_P - 1} \]

\[ \alpha_P - 1 = \frac{4\alpha_s N_c}{\pi} \ln 2 \]
The Main Course:

Small-x Evolution of the Quark Helicity Distribution
**Factorization:** One-to-one correspondence between the spin-dependent DIS cross-section and the quark helicity PDF.

\[
\frac{Q^2}{4\pi^2\alpha_{EM}} \frac{d\Delta\sigma^{(\gamma^* p)}}{dx \, dQ^2} = 2x \, g_1(x, Q^2) \overset{L.O.}{=} \sum_f e_f^2 \, x \Delta q_f(x, Q^2)
\]
• **Factorization:** One-to-one correspondence between the spin-dependent DIS cross-section and the quark helicity PDF.

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\]

\[
\frac{d \Delta \sigma^{(\gamma^* p)}}{dx \, dQ^2} \sim \left( \frac{1}{x} \right)^{\alpha_h - 1} \sim x \Delta q_f(x, Q^2)
\]
The spin-dependent DIS cross-section / hPDF is expressed in terms of a polarized dipole amplitude / cross section.

\[ x \Delta q^S(x, Q^2) \equiv \frac{Q^2}{2\pi^2 \alpha_{EM}} \sum_f \int \frac{d^2 x_{10} \, dz}{4\pi \, z (1-z)} \left[ |\Delta \Psi_{T,f}(x_{10}^2, z)|^2 + |\Delta \Psi_{L,f}(x_{10}^2, z)|^2 \right] \int d^2 b_{10} \frac{1}{zs} G_{10}(zs) \]
The spin-dependent DIS cross-section / hPDF is expressed in terms of a polarized dipole amplitude / cross section:

\[ x \Delta q^S(x, Q^2) \equiv \frac{Q^2 N_c}{2\pi^2 \alpha_{EM}} \sum_f \int \frac{d^2 x_{10} \, dz}{4\pi \, z(1-z)} \left[ |\Delta \Psi_{T,f}(x_{10}^2, z)|^2 + |\Delta \Psi_{L,f}(x_{10}^2, z)|^2 \right] \int d^2 b_{10} \frac{1}{zs} G_{10}(zs) \]

\[ \frac{1}{zs} G_{10}(zs) \equiv \left\langle \frac{1}{2N_c} \text{tr}[V_{x_0} V_{x_1}^\dagger] + c.c. \right\rangle_{(zs)} = -\frac{1}{4} \left( \frac{d \Delta \sigma^{(q_{x_0}^{unp} q_{x_1}^{pol})}}{d^2 b_{10}} (zs) + ch.c. \right) \]
The Polarized Dipole Amplitude

\[ \Delta q^S(x, Q^2) = \sum_f \int \frac{dr^-}{2\pi} e^{ixp^+r^-} \langle p| \bar{\psi}(0) U[0, r^-] \frac{\gamma^+\gamma^5}{2} \psi(r^-) |p \rangle \]

- The spin-dependent DIS cross-section / hPDF is expressed in terms of a polarized dipole amplitude / cross section

\[ x\Delta q^S(x, Q^2) = \frac{Q^2 N_c}{2\pi^2 \alpha_{EM}} \sum_f \int \frac{d^2 x_{10} \, dz}{4\pi z(1-z)} \left[ |\Delta \Psi_{T,f}(x_{10}^2, z)|^2 + |\Delta \Psi_{L,f}(x_{10}^2, z)|^2 \right] \int d^2 b_{10} \frac{1}{zs} G_{10}(zs) \]

\[ \frac{1}{zs} G_{10}(zs) = \left\langle \frac{1}{2N_c} \text{tr} \left[ V_{x_0}^{\text{pol}} V_{x_1}^{\dagger} \right] + c.c. \right\rangle_{(zs)} = -\frac{1}{4} \left( \frac{d \sigma^{(q_{x_0}^{\text{unp}} q_{x_1}^{\text{pol}})}}{d^2 b_{10}}(zs) + c.h.c. \right) \]

"Polarized Wilson line"
Helicity Evolution: Initial Conditions

- Initial conditions from the quark target model
  - Quark and sub-eikonal gluon exchange

\[
G_{10}^{(0)}(zs) = \frac{\alpha_s^2 C_F}{N_c} \left[ \frac{C_F}{x_{1T}^2} - 2\pi \delta^2(x_1) \ln(zs x_{10}^2) \right]
\]
Helicity Evolution: Initial Conditions

- Initial conditions from the quark target model
  - Quark and sub-eikonal gluon exchange

\[ G_{10}^{(0)}(zs) = \frac{\alpha_s^2 C_F}{N_c} \left[ \frac{C_F}{x_{1T}^2} - 2\pi\delta^2(x_1) \ln(zs x_{10}^2) \right] \]

- Polarization transfer is suppressed at small x
  - Leading term has exactly 1 sub-eikonal exchange
Helicity Evolution: Initial Conditions

- **Initial conditions** from the quark target model
  - Quark and sub-eikonal gluon exchange

\[ G_{10}^{(0)}(zs) = \frac{\alpha_s^2 C_F}{N_c} \left[ \frac{C_F}{x_1^2} - 2\pi \delta^2(x_1) \ln(zs x_{10}^2) \right] \]

- Polarization transfer is suppressed at small x
  - Leading term has exactly 1 sub-eikonal exchange

\[ \frac{d \Delta \sigma^{(q_{x0}^{unp} q_{x1}^{pol})}}{d^2 b_{10}}(zs) \propto \frac{1}{zs} \]

- Include this known scaling in the definition of the polarized dipole amplitude

\[ \frac{1}{zs} G_{10}(zs) \equiv \left\langle \frac{1}{2N_c} \text{tr}[V_{x0} V_{x1}^{pol \dagger}] + c.c. \right\rangle (zs) = -\frac{1}{4} \left( \frac{d \Delta \sigma^{(q_{x0}^{unp} q_{x1}^{pol})}}{d^2 b_{10}}(zs) + \text{ch.c.} \right) \]
• Soft polarized quark and gluon emission spans the full rapidity interval:

\[
\int \frac{dz'}{z'} \to \ln \frac{z_8}{\Lambda^2}
\]

• The polarized line is also sensitive to collinear / short-distance fluctuations:

\[
\int \frac{dx^2_{21}}{x^2_{21}} \to \ln \frac{z_8}{\Lambda^2}
\]
Origins of Helicity Evolution

- Soft polarized quark and gluon emission spans the full rapidity interval:

\[
\int \frac{dz'}{z'} \rightarrow \ln \frac{z_2}{\Lambda^2}
\]

- The polarized line is also sensitive to collinear / short-distance fluctuations:

\[
\langle V_1^{\text{pol} \dagger}(z_1) \rangle \sim \int \frac{dz_2}{z_2} \int d^2 x_2 \left( \frac{\alpha_s C_F}{2\pi^2} \frac{z_2}{z_1} \frac{1}{x_{21}^2} \right) \langle V_2^{\text{pol} \dagger}(z_2) \rangle \\
\sim \frac{1}{z_1 s} \\
G_{10}(z_1) \sim \frac{\alpha_s C_F}{2\pi} \int \frac{dz_2}{z_2} \int \frac{dx_{21}^2}{x_{21}^2} \ G_{21}(z_2) \\
\ln^2 \frac{z_1 s}{\Lambda^2}
\]

\[
\int \frac{dx_{21}^2}{x_{21}^2} \rightarrow \ln \frac{z_2}{\Lambda^2}
\]
• Soft polarized quark and gluon emission spans the full rapidity interval:

\[ \int \frac{dz'}{z'} \to \ln \frac{z_s}{\Lambda^2} \]

• The polarized line is also sensitive to collinear / short-distance fluctuations:

\[ \left\langle V_{1pol}^\dagger(z_1) \right\rangle \sim \int \frac{dz_2}{z_2} \int d^2x_2 \left( \frac{\alpha_s C_F}{2\pi^2} \frac{z_2}{z_1} \frac{1}{x_{21}^2} \right) \left\langle V_{2pol}^\dagger(z_2) \right\rangle \sim \frac{1}{z_1 s} \]

\[ G_{10}(z_1) \sim \frac{\alpha_s C_F}{2\pi} \int \frac{dz_2}{z_2} \int \frac{dx_{21}^2}{x_{21}^2} G_{21}(z_2) \]

\[ \ln^2 \frac{z_1 s}{\Lambda^2} \]

• Helicity evolution is double logarithmic
  ➢ Stronger than unpolarized evolution!

\[ \alpha_s \ln^2 \frac{1}{x} \sim 1 \]
An Example: The Collinear BFKL Sector

- No collinear logarithms

\[ \sim \int d^2 x_2 \frac{(x_{21} \cdot x_{20})}{x_{21}^2 x_{20}^2} \]
An Example: The Collinear BFKL Sector

\[ \int d^2 x_2 \frac{(x_{21} \cdot x_{20})}{x_{21}^2 x_{20}^2} \]

- No collinear logarithms

\[ \int d^2 x_2 \frac{1}{x_{20}} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_2} V_{x_1}^{\text{pol} \dagger}] \text{tr}[V_{x_0} V_{x_2}^{\dagger}] \right\rangle \right. \\
\left. - \frac{1}{N_c} \left\langle \text{tr}[V_{x_0} V_{x_1}^{\text{pol} \dagger}] \right\rangle \right] \]

- Collinear enhancement as \( x_2 \to x_0 \), but vanishing support due to real-virtual cancellations
An Example: The Collinear BFKL Sector

\[ \sim \int d^2 x_2 \frac{(x_{21} \cdot x_{20})}{x_{21}^2 x_{20}^2} \]

- No collinear logarithms

\[ \sim \int d^2 x_2 \frac{1}{x_{20}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_2} V_{x_1}^{pol \uparrow}] \text{tr}[V_{x_0} V_{x_2}^{\uparrow}] \right\rangle \right. \\
- \frac{1}{N_c} \left\langle \text{tr}[V_{x_0} V_{x_1}^{pol \uparrow}] \right\rangle \]

- Collinear enhancement as \( x_2 \to x_0 \), but vanishing support due to real-virtual cancellations

\[ \sim \int d^2 x_2 \frac{1}{x_{21}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_2} V_{x_1}^{pol \uparrow}] \text{tr}[V_{x_0} V_{x_2}^{\uparrow}] \right\rangle \right. \\
- \frac{1}{N_c} \left\langle \text{tr}[V_{x_0} V_{x_1}^{pol \uparrow}] \right\rangle \]

- Collinear enhancement as \( x_2 \to x_1 \), about the distinct polarized line
• Because one line of the dipole is polarized, a subset of the BFKL kernel becomes double logarithmic (DLA) for $x_{21}^2 \ll x_{10}^2$

$$\delta G_{10}(zs) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int \frac{d^2 x_{21}}{x_{21}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_2} V_{x_1}^{\text{pol}}] \text{tr}[V_{x_0} V_{x_2}^\dagger] \right\rangle_{(z's)} - \frac{1}{N_c} \left\langle \text{tr}[V_{x_0} V_{x_1}^{\text{pol}}] \right\rangle_{(z's)} \right]$$

$$\frac{1}{2} \ln^2(zs x_{10}^2)$$
Because one line of the dipole is polarized, a subset of the BFKL kernel becomes double logarithmic (DLA) for \( x_{21}^2 \ll x_{10}^2 \)

\[
\delta G_{10}(z \sigma) = \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{1}{z_{10}^2}}^{z} \frac{dz'}{z'} \int_{\frac{1}{x_{21}^2}}^{x_{10}^2} \frac{d^2 x_{21}}{x_{21}^2} \left[ \frac{1}{N_c^2} \left\langle \operatorname{tr}[V_{x_2} V_{x_1}^{\text{pol} \dagger}] \operatorname{tr}[V_{x_0} V_{x_2}^\dagger] \right\rangle_{(z' \sigma)} - \frac{1}{N_c} \left\langle \operatorname{tr}[V_{x_0} V_{x_1}^{\text{pol} \dagger}] \right\rangle_{(z' \sigma)} \right]
\]

\[
\frac{1}{2} \ln^2(z \sigma x_{10}^2)
\]

To continue generating both soft and collinear logs, further evolution must be doubly ordered:

\[
z \gg z' \gg z'' \gg \cdots
\]

\[
z \Delta x_T^2 \gg z' \Delta x_T'^2 \gg z'' \Delta x_T''^2 \gg \cdots
\]
New Contributions: Polarized Quarks

- Emission of a soft polarized (anti)quark is only possible from the polarized line
  - DLA contribution extends over the whole ordered phase space
  - No $x_{21}^2 \ll x_{10}^2$ restriction
New Contributions: Polarized Quarks

- Emission of a **soft polarized (anti)quark** is only possible from the polarized line
  - DLA contribution extends over the whole ordered phase space
  - No $x_{21}^2 \ll x_{10}^2$ restriction

\[
\delta G_{10}(zs) = \frac{\alpha_s N_c}{4\pi^2} \int_{\frac{z}{z'}}^{z} \int_{\frac{1}{z'}^{z'}} \frac{x_{10}^2 z'}{x_{21}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr}[V_{x_1} V^\text{pol} V_{x_2}] \right\rangle (z's) - \frac{1}{N_c^3} \left\langle \text{tr}[V_{x_0} V^\text{pol} V_{x_2}] \right\rangle (z's) \right] \]

\[
\ln(zs x_{10}^2) \ln(zs/\Lambda^2)
\]

M. Sievert

Quark Helicity Evolution at Small $x$
New Contributions: Polarized Gluons

• Emission of a soft polarized gluon can couple to both lines

\[ \delta G_{10}(z_s) = + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int \frac{d^2 x_{21}}{x_{21}^2} \left[ \frac{1}{N_c^2} \left( \text{tr} \left[ t^b V_{x_0} t^a V_{x_1}^\dagger \right] \left( U_{x_2}^{\text{pol}} \right)^{ba} \right) \right] (z's) \]

Ladder (from polarized line only): DLA everywhere
New Contributions: Polarized Gluons

- Emission of a soft polarized gluon can couple to both lines

**Ladder (from polarized line only):**

\[
\delta G_{10}(zs) = + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2 x_{21} \frac{x_{10}^2 z'}{x_{21}^2} \left[ \frac{1}{N_c^2} \left\langle \text{tr} \left[ t^b V_{x_0} t^a V_{x_1}^\dagger \right] \left( U_{x_2}^{\text{pol}} \right)^{ba} \right\rangle \right]_{(z's)}
\]

**Non-Ladder (across the dipole):**

\[
\delta G_{10}(zs) = - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2 x_{21} \left( \frac{x_{21} \cdot x_{20}}{x_{21}^2 x_{20}^2} \right) \left[ \frac{1}{N_c^2} \left\langle \text{tr} \left[ t^b V_{x_0} t^a V_{x_1}^\dagger \right] \left( U_{x_2}^{\text{pol}} \right)^{ba} \right\rangle \right]_{(z's)}
\]

Cancel ladders for \( x_{21}^2 \gg x_{10}^2 \)
New Contributions: Polarized Gluons

- Emission of a **soft polarized gluon** can couple to both lines

- **Ladder** (from polarized line only): **DLA everywhere**

\[
\delta G_{10}(zs) = + \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int \frac{d^2x_{21}}{x_{21}^2} \frac{x_{10}^2}{x_{10}^2} \left[ \frac{1}{N_c^2} \langle \text{tr}[t^b V_0 \cdot t^a V_1] (U_{x_2}^{pol})^{ba} \rangle \right]_{(z's)}
\]

- **Non-Ladder** (across the dipole): **Cancel ladders for** \(x_{21}^2 \gg x_{10}^2\)

\[
\delta G_{10}(zs) = - \frac{\alpha_s N_c}{2\pi^2} \int \frac{dz'}{z'} \int \frac{d^2x_2}{x_{21}^2} \frac{x_{21}^2}{x_{20}^2} \left[ \frac{1}{N_c^2} \langle \text{tr}[t^b V_0 \cdot t^a V_1] (U_{x_2}^{pol})^{ba} \rangle \right]_{(z's)}
\]

- **Limits the DLA phase space** for polarized ladder gluons: \(x_{21}^2 \ll x_{10}^2\)

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Quark Helicity Evolution at Small x  
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• DLA evolution is only sensitive to fluctuations near the polarized line

➢ For all gluon emissions, ladder / non-ladder cancellation limits the DLA to the strongly ordered phase space $x_{21}^2 \ll x_{10}^2$
• DLA evolution is only sensitive to fluctuations near the polarized line

> For all gluon emissions, ladder / non-ladder cancellation limits the DLA to the strongly ordered phase space \(x_{21}^2 \ll x_{10}^2\)

• The polarized analog of BFKL is the linearized, large-\(N_c\) limit
• DLA evolution is only sensitive to fluctuations near the polarized line

- For all gluon emissions, ladder / non-ladder cancellation limits the DLA to the strongly ordered phase space $x_{21}^2 \ll x_{10}^2$

• The polarized analog of BFKL is the linearized, large-$N_c$ limit

- There is a nontrivial constraint on the lifetime of large polarized dipoles from their short-lived “neighbor” fluctuations!
The impact-parameter integrated dipole amplitude evolves as:

\[
G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2} \left[ \Gamma(x_{10}^2, x_{21}^2, z') + 3G(x_{21}^2, z') \right]
\]

\[
\Gamma(x_{10}^2, x_{21}^2, z') = G^{(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{2\pi} \int \frac{dz''}{z''} \int \frac{dx_{32}^2}{x_{32}^2} \left[ \Gamma(x_{10}^2, x_{32}^2, z'') + 3G(x_{32}^2, z'') \right]
\]
Up Next: Dessert

Finding a Solution at Small x
The Task at Hand

\[
G(x^2_{10}, z) = G^{(0)}(x^2_{10}, z) + \frac{\alpha_s N_c}{2\pi} \int_{x^2_{10}s}^{z} \frac{dz'}{z'} \int_{x^2_{21}s}^{x^2_{10}} \frac{dx^2_{21}}{x^2_{21}} \left[ \Gamma(x^2_{10}, x^2_{21}, z') + 3G(x^2_{21}, z') \right]
\]

\[
\Gamma(x^2_{10}, x^2_{21}, z') = G^{(0)}(x^2_{10}, z') + \frac{\alpha_s N_c}{2\pi} \int_{x^2_{10}s}^{z'} \frac{dz''}{z''} \int_{x^2_{32}s}^{x^2_{10}} \frac{dx^2_{32}}{x^2_{32}} \left[ \Gamma(x^2_{10}, x^2_{32}, z'') + 3G(x^2_{32}, z'') \right]
\]

- Must solve a set of coupled integro-differential equations
  - Neighbor dipole depends on two spatial arguments
The Task at Hand

\[ G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2} \left[ \Gamma(x_{10}^2, x_{21}^2, z') + 3G(x_{21}^2, z') \right] \]

\[ \Gamma(x_{10}^2, x_{21}^2, z') = G^{(0)}(x_{10}^2, z') + \frac{\alpha_s N_c}{2\pi} \int \frac{dz''}{z''} \int \frac{dx_{32}^2}{x_{32}^2} \left[ \Gamma(x_{10}^2, x_{32}^2, z'') + 3G(x_{32}^2, z'') \right] \]

- Must solve a set of coupled integro-differential equations
  - Neighbor dipole depends on two spatial arguments

- Can it be done numerically? Can it be done analytically?
  - What is the helicity intercept at small x?

\[ \frac{d \Delta \sigma}{dx \ dQ^2} \sim \left( \frac{1}{x} \right)^{\alpha_h - 1} \sim x \Delta q_f(x, Q^2) \]

\[ \alpha_h = ??? \]
Conclusion: A Taste of the Answer

- Quark helicity at small $x$ receives strong double-logarithmic enhancement through the evolution near a polarized Wilson line

adapted from Aschenauer et al., Phys. Rev. D92 (2015) no.9 094030