Analyticity in Spin and Causality in Conformal Theories

(based on 1703.00278)

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[Nuclear Theory]/RIKEN seminar, Brookhaven, April 18th 2017

Outline

The aim of this talk will be to present a formula…

1. Why conformal Regge theory?
   -Conformal bootstrap and large spin physics
   -AdS/CFT and bulk locality

2. CFT Froissart-Gribov formula
   -Why operators are analytic in spin
   -New ingredients in CFT (ANEC&bound on chaos)

3. Applications:
   -operators of large spin
   -CFTs dual to gravity: causality&bulk locality

Conformal bootstrap

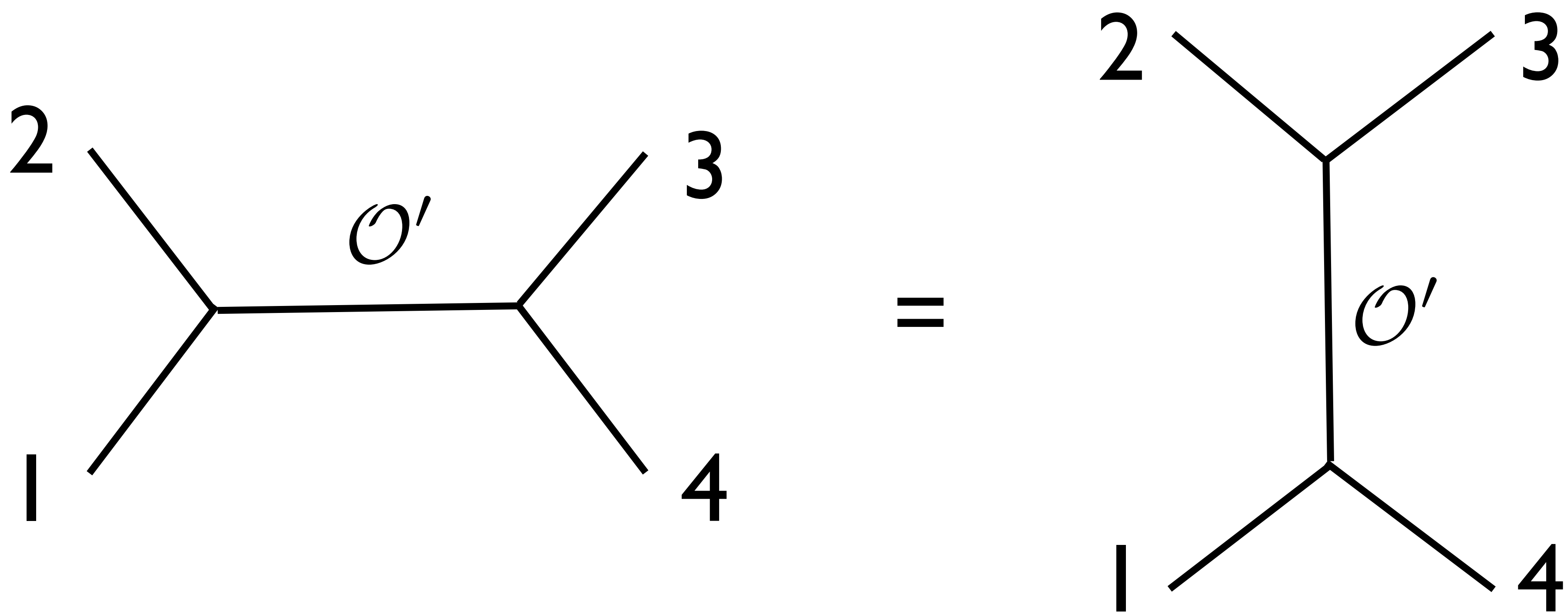
- Input: Operator Product Expansion

$$\lim_{y\to x} \mathcal{O}(x)\mathcal{O}(y) = \sum_{\mathcal{O}'} f_{\mathcal{O}\mathcal{O}\mathcal{O}'} (x-y)^{\#} \mathcal{O}'(x) \quad (+\text{derivatives})$$

- Converges in finite radius
- In CFT, operator have scaling exponents:

$$\# = \Delta_{\mathcal{O}'} - 2\Delta_{\mathcal{O}}$$

- Dynamics: Crossing Equation



$$\sum_{\mathcal{O}'} f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'} (x_{12} x_{34})^{\#} = \sum_{\mathcal{O}'} f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'} (x_{23} x_{14})^{\#}$$

(+derivatives) (+derivatives)

Why is it constraining?

Imagine points are slightly closer to one limit:

$$\sum_{\mathcal{O}'} f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}(\text{smaller}) = \sum_{\mathcal{O}'} f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}(\text{bigger})$$

$$\Rightarrow\ 0 = \sum_{\mathcal{O}'} f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}(\text{bigger} - \text{smaller})$$

Naively, no solution!! (since f^2's are positive)

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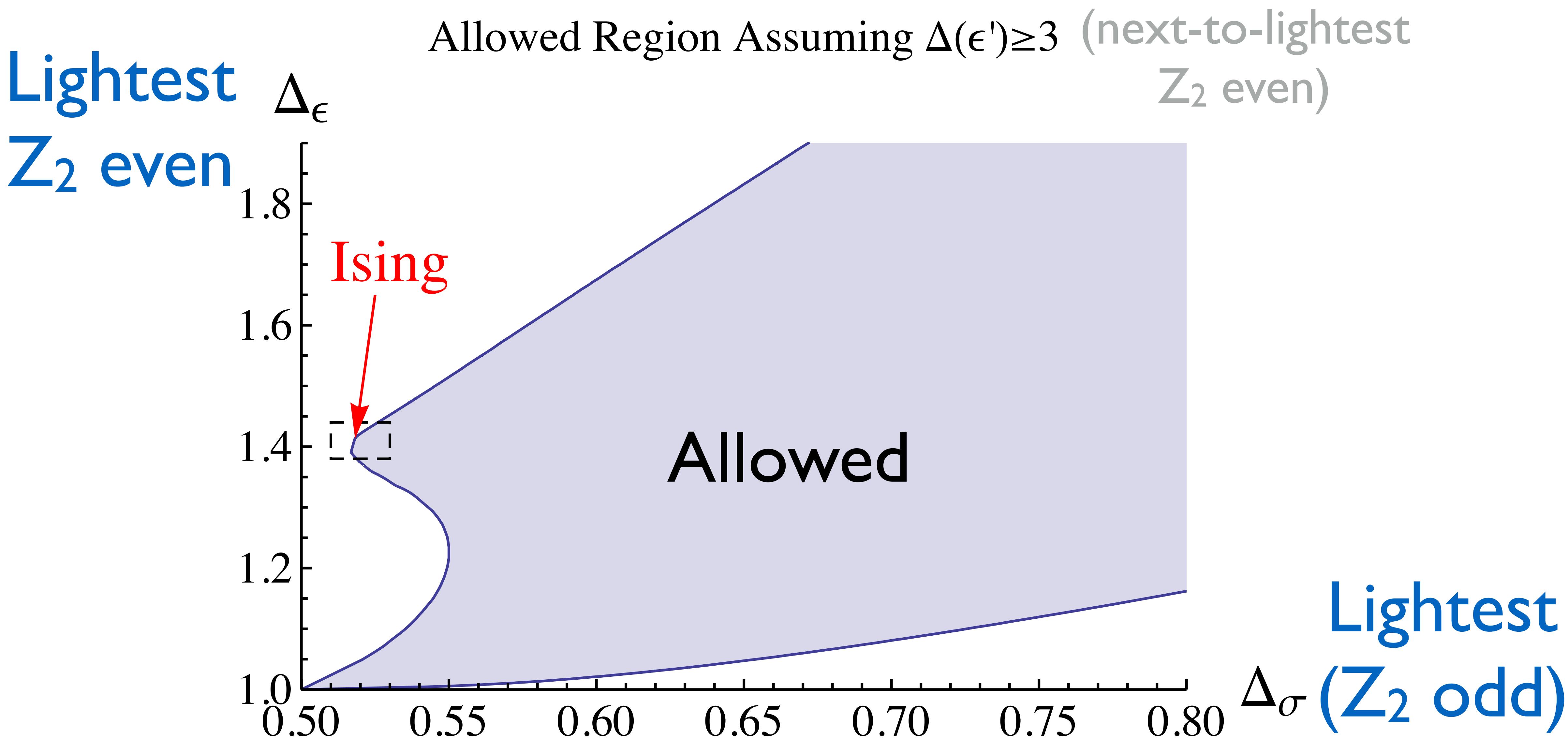
Naively, no solution!! (since f^2's are positive)

But: first few terms go the other way: $(\# = \Delta_{\mathcal{O}'} - 2\Delta_{\mathcal{O}})$

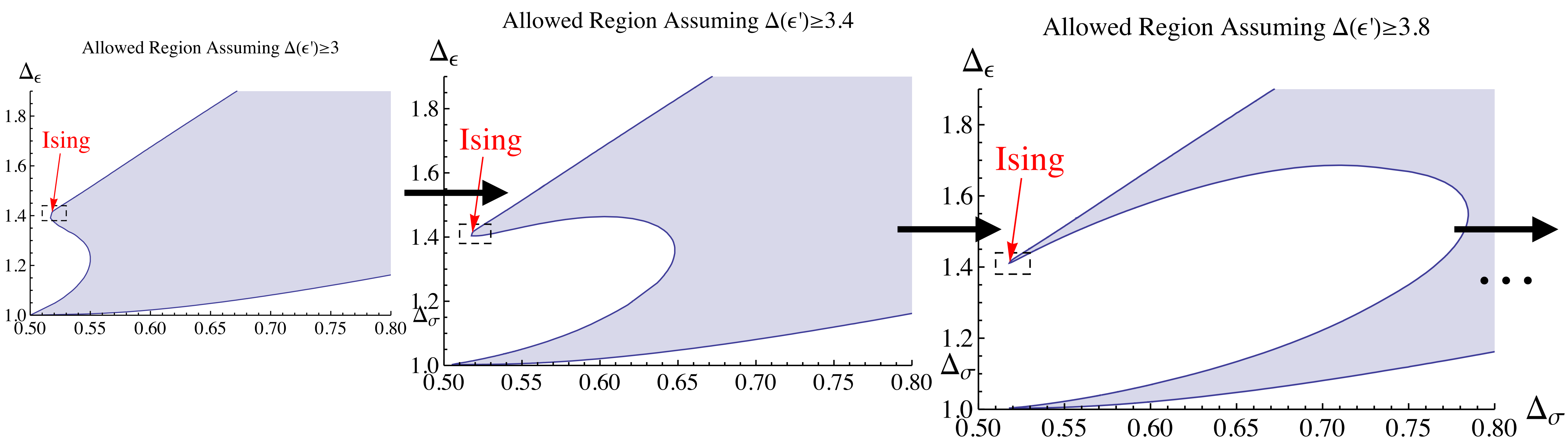
$$\Rightarrow \sum_{\text{first few}} f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}(\text{positive}) = \sum_{\text{rest}} f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}(\text{positive})$$

Numerical exclusion plots: the 3D Ising CFT



[From: El-Showk,Paulos,Poland, Rychkov,Simmons-Duffin&Vichi '12]

As one learns more about next-to-lightest op, an island is carved out



The `tip' which can't be excluded, must be the theory we're looking for!

Computer algorithm to solve multiple inequalities:
semidefinite programming [Poland,Simmons-Duffin&Vichi '11]

Leads to precise, quantitative critical exponents, in various CFTs. Ex: 3D Ising:

spin & $\mathbb{Z}_2$	name	Δ	OPE coefficient
$\ell = 0$, $\mathbb{Z}_2 = -$	σ	$0.518154(15)$	
$\ell = 0$, $\mathbb{Z}_2 = +$	ϵ	$1.41267(13)$	$f^2_{\sigma\sigma\epsilon} = 1.10636(9)$
	ϵ'	$3.8303(18)$	$f^2_{\sigma\sigma\epsilon'} = 0.002810(6)$
$\ell = 2$, $\mathbb{Z}_2 = +$	T	3	$c/c_{\text{free}} = 0.946534(11)$
	T'	$5.500(15)$	$f^2_{\sigma\sigma T'} = 2.97(2) \times 10^{-4}$

[El-Showk,Paulos,Poland, Rychkov,Simmons-Duffin&Vichi '14]

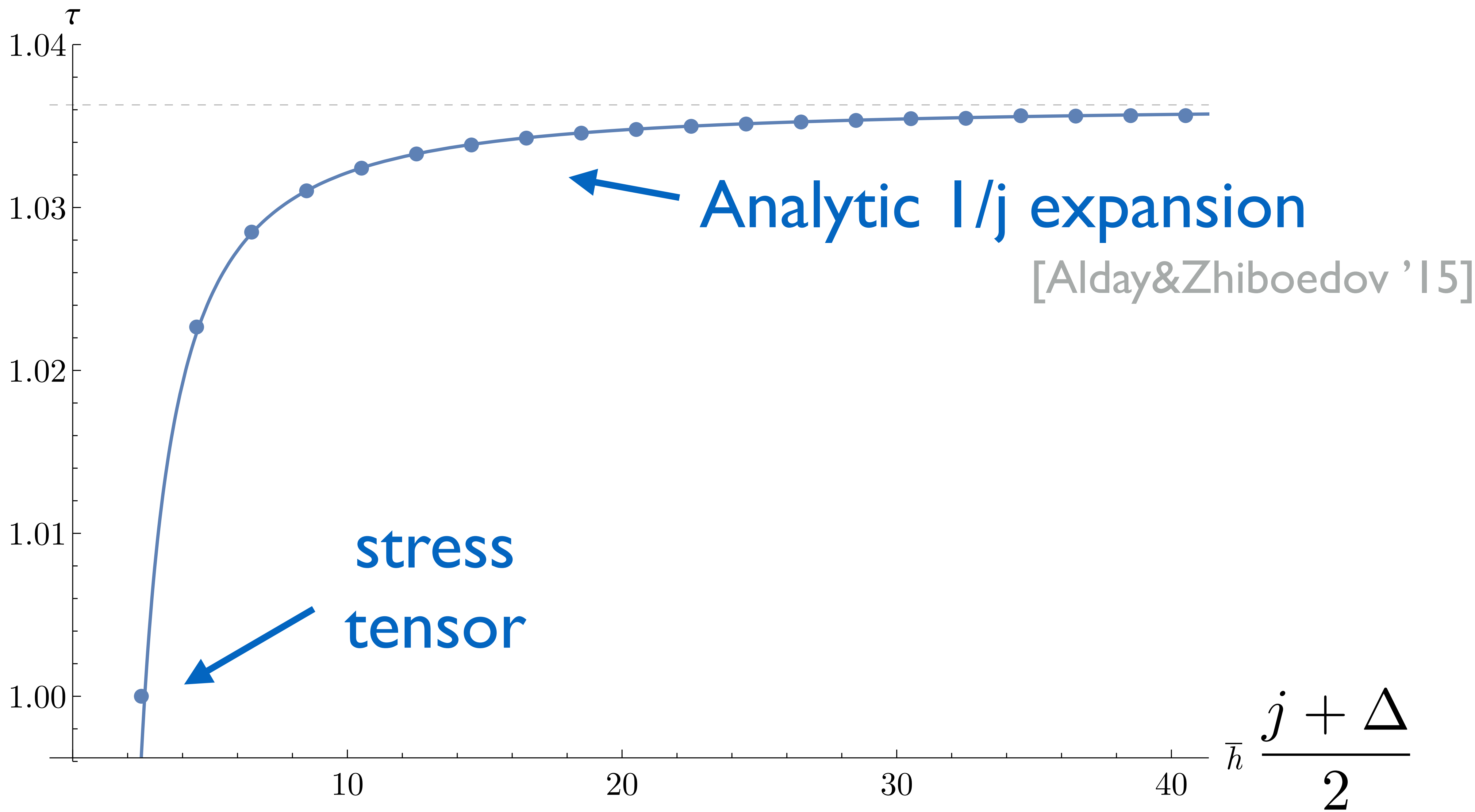
Why CFTs?

- CFTs are interesting:
  - Critical exponents in phase transitions
  - Many interesting theories are near-conformal (e.g. QCD at high energies)
  - Any theory of gravity in AdS is dual to a CFT

- CFTs are simpler:
  - 4-pt function depends on only 2 cross-ratios (compare with 6 distances: x_{ij}^2/ℓ_0^2 !)
  - total derivatives not independent operators

Empirical observation: operators lie in smooth families

$$\tau_{[\sigma\sigma]_0}(\bar{h}) \qquad = \Delta - j$$



$$\frac{j+\Delta}{2}$$

[Plot from Simmons-Duffin '16]

Why Conformal Regge Theory?

To explain why operators organize into families ('Regge trajectories')

Quantitatively, a Froissart-Gribov inversion formula:

$$f^2_{\mathcal{O}\mathcal{O}\mathcal{O}'} = \int dz d\bar{z}\, (\ldots)\, \text{``Im}\, \mathcal{M}\text{''}$$

In AdS/CFT, this formula will know about bulk locality !

Second motivation

Conjecture:

Any large-N CFT with a large gap of operator dimension has an AdS dual, down to lengths $\ell_{\rm AdS}/\Delta_{\rm gap}$

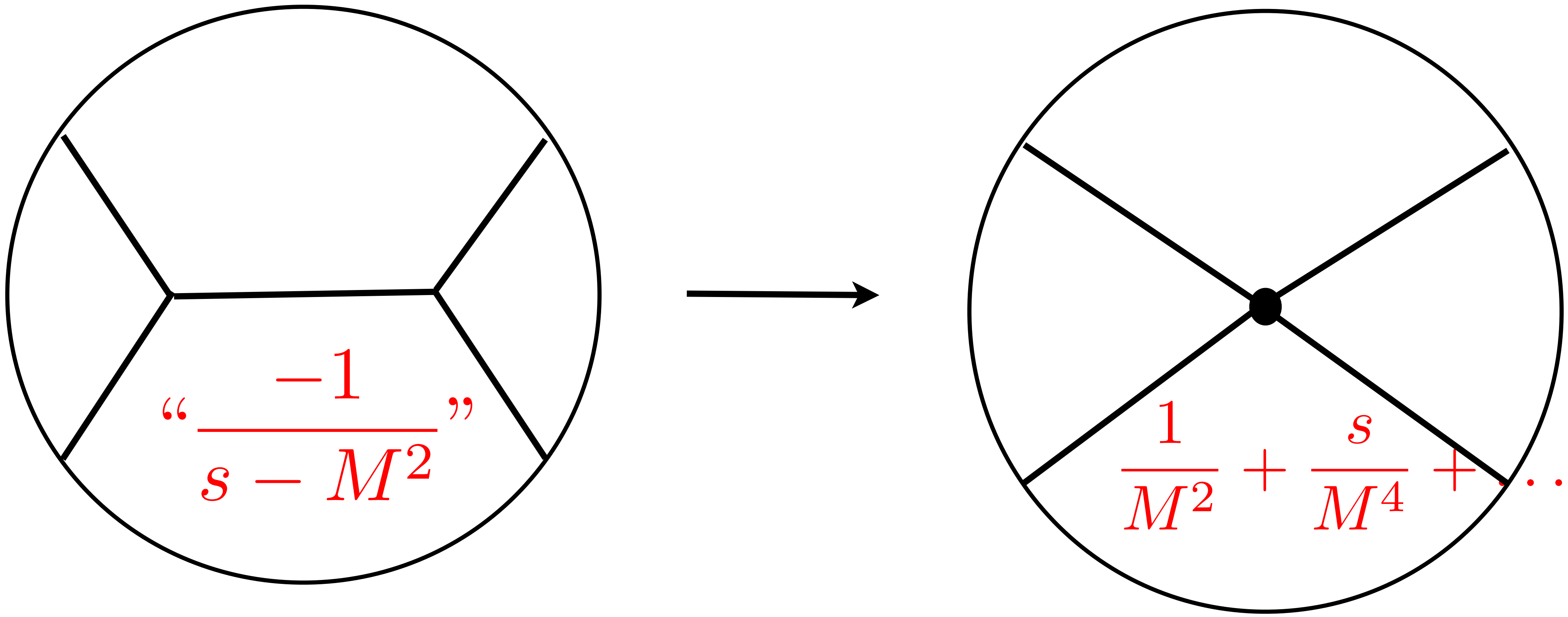
[Heemskerk,Penedones,Polchinski& Sully '09]

They proved: solutions to crossing in large-N CFTs w/gap

$\longleftrightarrow$ local interactions in AdS

But why are higher-dim interactions suppressed by powers of $\Delta_{\rm gap}$?

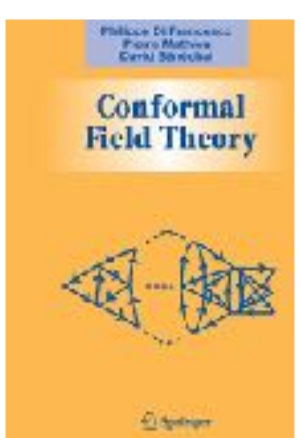
Effective field theory in AdS:



In string theory, for example, $M \sim M_{\text{string}} \gg 1/R_{\text{AdS}}$

Suppression would be clear from a ‘dispersion relation in the flat space limit of AdS’:

$$\mathcal{M}(s) \sim \int_{M^2}^{\infty} \frac{ds'}{s' - s} \mathrm{Im}\mathcal{M}(s') \sim \frac{1}{M^2} + \frac{s}{M^4} + \dots$$



Sigh, if only a CFT formula existed that read like this!

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Why is physics analytic in spin?

Short answer:

because Euclidean physics has to resum into something sensible at high energies

- Toy model: single-variable power series

$$f(x) = \sum_{j=1}^{\infty} f_j x^j \qquad \text{('Euclidean OPE')}$$

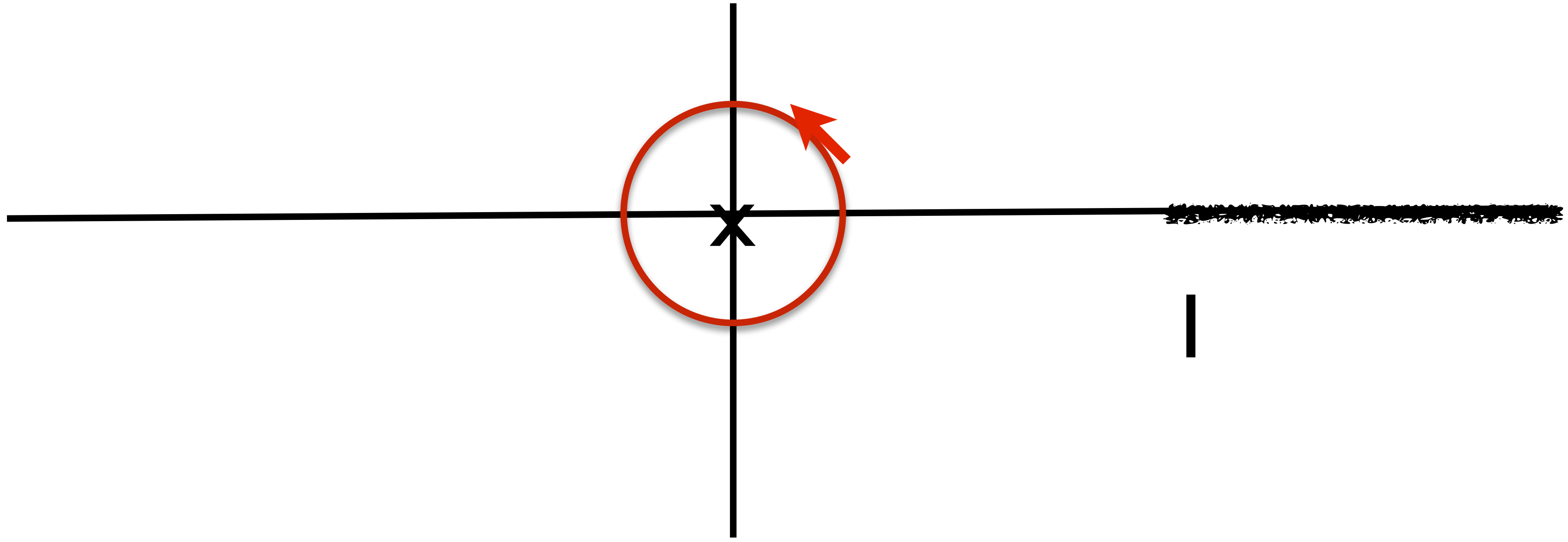
- Assume:
  - $f(x)$ is analytic in cut plane $\mathbb{C}\backslash[1,\infty)$
  - $|f(x)/x| \to 0$ at infinity

(large x= 'large energy')

⇒ What does this tell us about the f_j's?

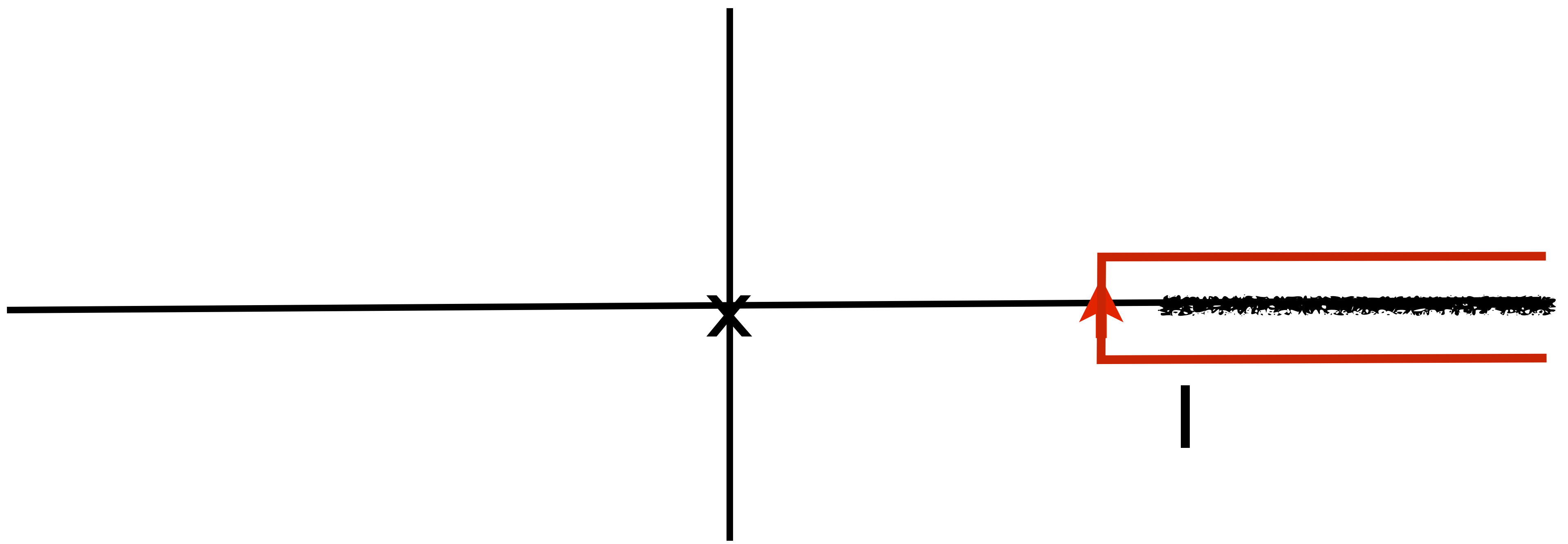
- Q: How to extract f_j from f(x) ?
- A: Cauchy

$$f_j(x) = \frac{1}{2\pi i} \oint \frac{dx}{x} \, x^{-j} \, f(x)$$



- Q: How to extract f_j from $f(x)$?
- A: Cauchy

Deform the contour:



$$\Rightarrow \quad f_j = \frac{1}{2\pi i}\int_1^\infty \frac{dx}{x} x^{-j}\,\mathrm{Disc}\, f(x) \qquad (j \geq 1)$$

Basic inversion formula

$$f_j = \int_1^\infty \frac{dx}{x} x^{-j} \frac{\mathrm{Disc}\, f(x)}{2\pi i}$$

(Sanity check: $f(x) = -\log(1-x) \Rightarrow \frac{\mathrm{Disc}\, f(x)}{2\pi i} = 1$)

$$\Rightarrow f_j = 1/j \quad ✔$$

⇒ Good high-energy behavior leads to:

-Taylor coefficients are analytic for Re[j]≥1 !

-Determined by imaginary part of amplitude

Froissart-Gribov formula

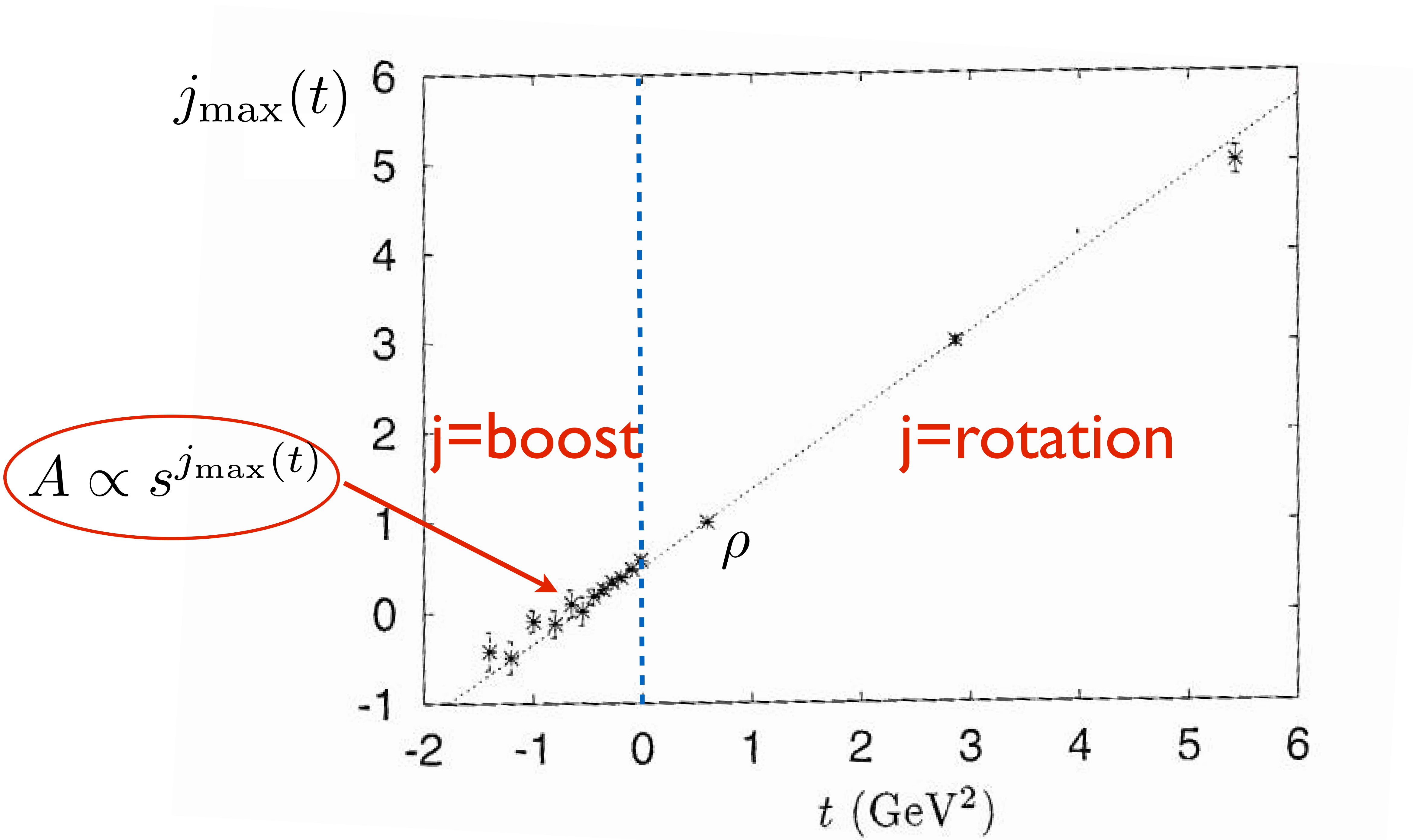
inverts Legendre polynomials:

$$A(s, \cos\theta) = \sum_{j=0}^{\infty} a_j(s) P_j(\cos\theta)$$

$$\Leftrightarrow a_j^{(t)} = \int_{\eta_0}^{\infty} d\eta \, Q_j(\cosh\eta) \, \text{Disc}\, A(\cosh(\eta)) + (-1)^j [\text{t-channel}]$$

- Explains why S-matrix can be decomposed into analytic-in-spin partial waves
- Foundation of Regge theory

Classic application to QCD



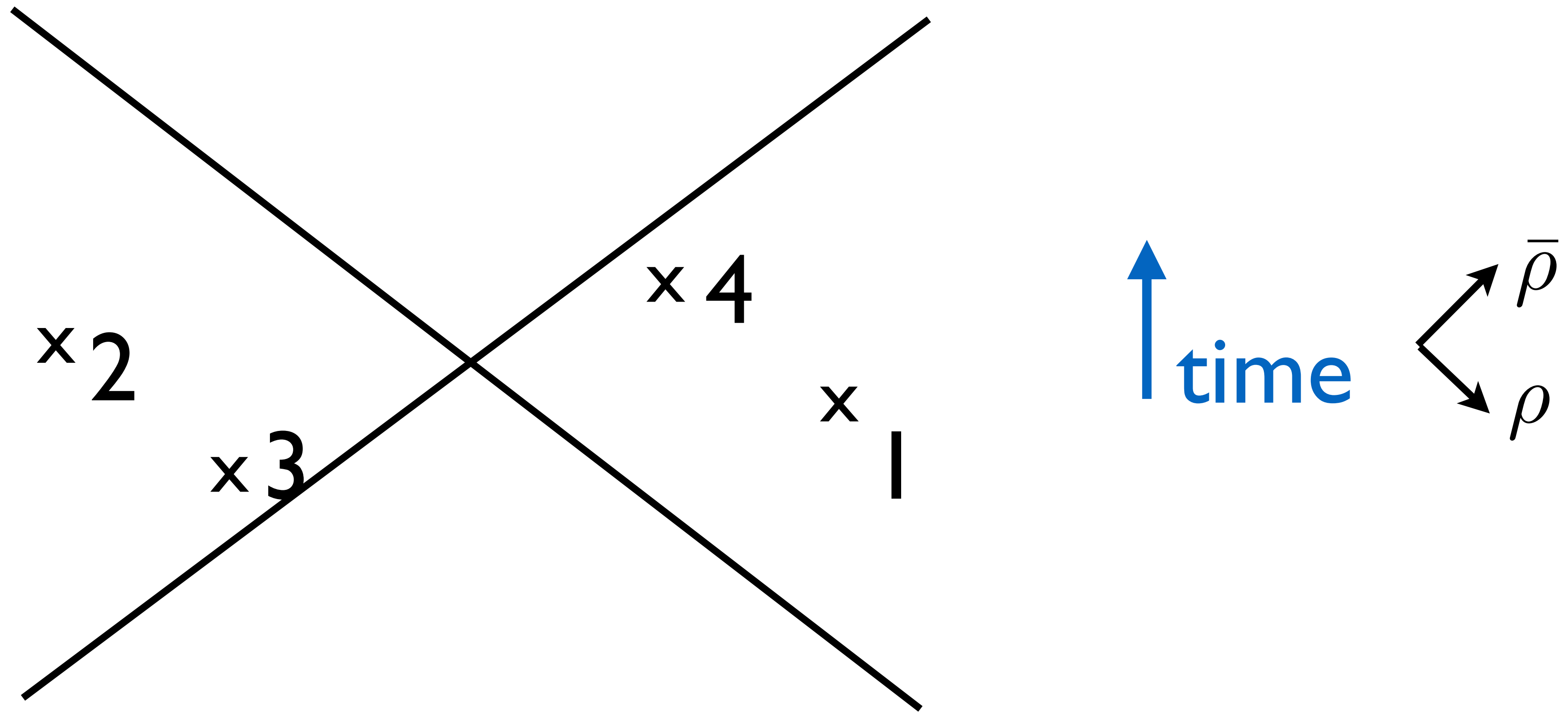
t<0 obtained from $\pi^- p \to \pi^0 n$ data, (@3.6,5.85&13.3GeV/c)

[from Donnachie, Dosch,Landshoff&Nachtmann]

Suppose we had a Froissart-Gribov formula in CFT

What should be ‘$\mathrm{Im}\, \mathcal{M}$’ ?

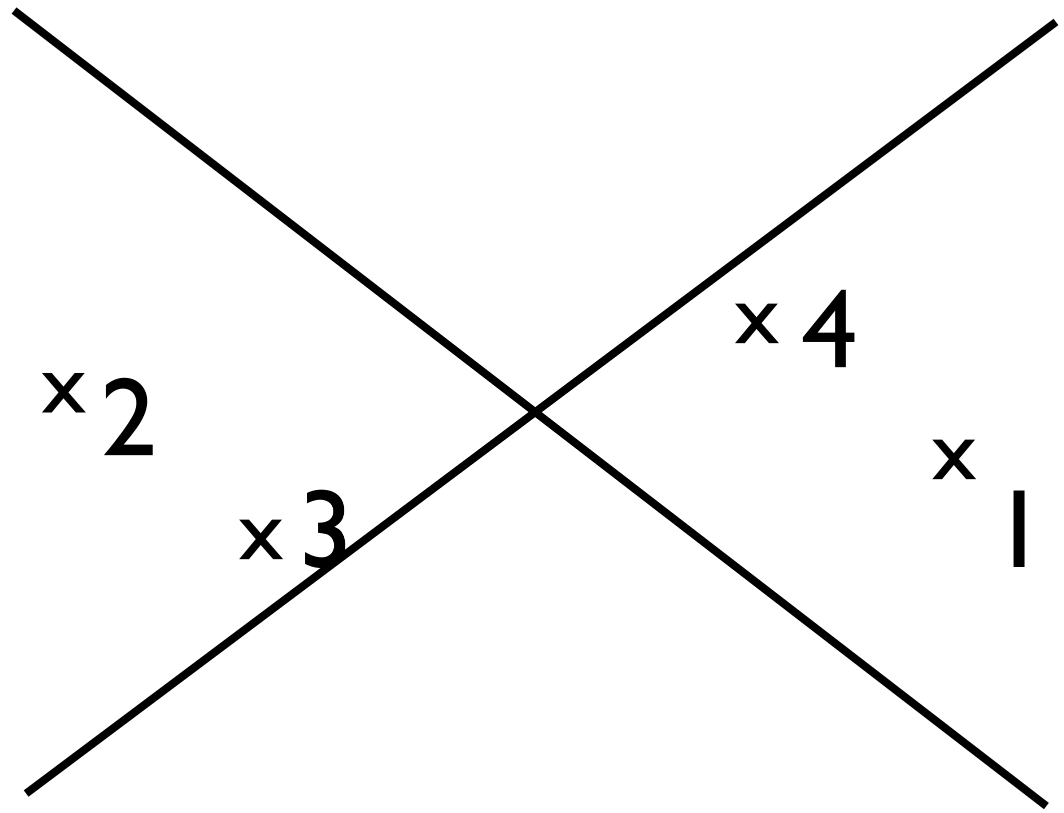
- We consider 4-point correlator in CFT$_d$



- Symmetrical param. within Rindler wedges:

$$-\rho_2 = \rho_1 = 1$$
$$-\rho_3 = \rho_4 \equiv \rho$$

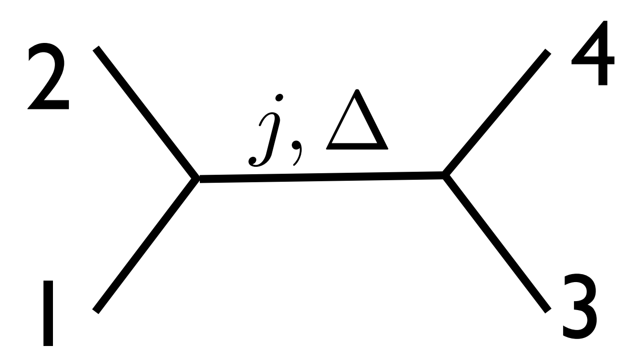
- We consider 4-point correlator in CFT_d



$$-\rho_2 = \rho_1 = 1$$
$$-\rho_3 = \rho_4 \equiv \rho$$

- at small ρ, s-channel OPE:

$$G(\rho, \bar{\rho}) = \sum_{j,\Delta} c_{j,\Delta} \rho^{\frac{\Delta - j}{2}} \bar{\rho}^{\frac{\Delta + j}{2}} =$$



(one normally replaces power series by 'blocks' which include derivatives of primary operators:

$$G_{J,\Delta}(z,\bar{z}) = \frac{k_{\Delta-J}(z)k_{\Delta+J}(\bar{z}) + k_{\Delta+J}(z)k_{\Delta-J}(\bar{z})}{1+\delta_{J,0}} \qquad (d=2)\,,$$

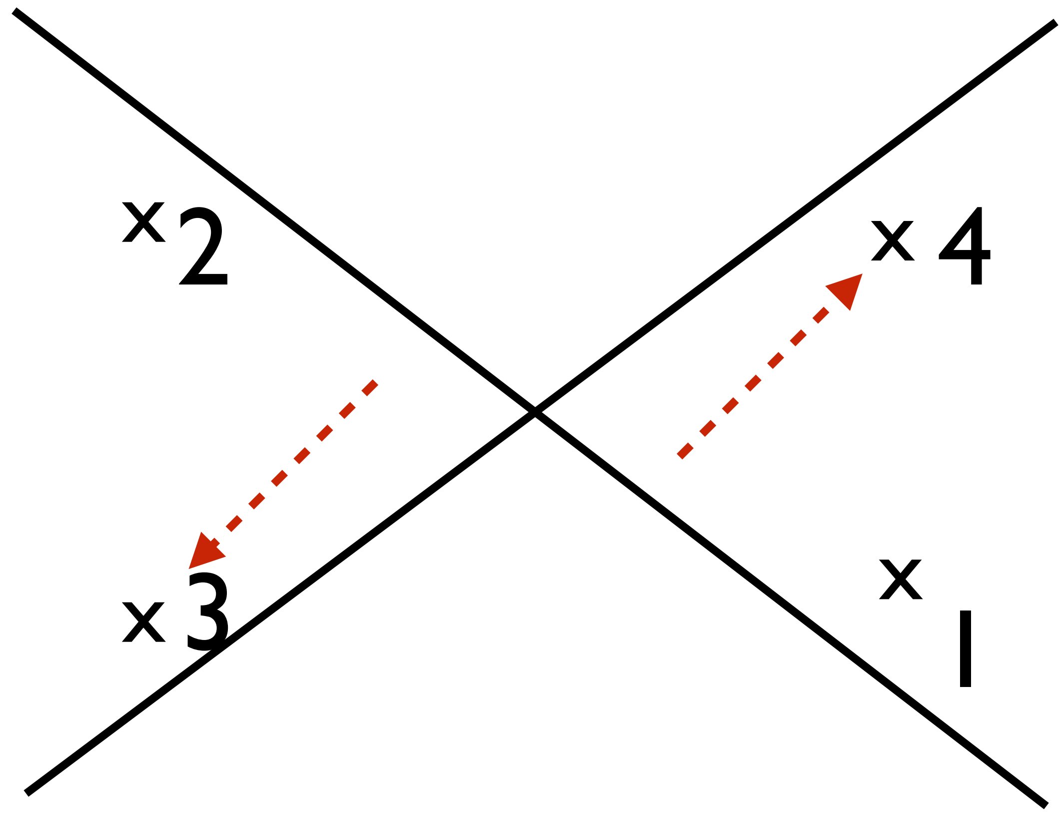
$$G_{J,\Delta}(z,\bar{z}) = \frac{z\bar{z}}{\bar{z}-z}\left[k_{\Delta-J-2}(z)k_{\Delta+J}(\bar{z}) - k_{\Delta+J}(z)k_{\Delta-J-2}(\bar{z})\right] \qquad (d=4)\,.$$

where

$$k_\beta(z) = z^{\beta/2}{}_2F_1(\beta/2,\beta/2,\beta,z) \qquad z = \frac{4\rho}{(1+\rho)^2}$$

This will be important below,
but Taylor series will suffice for now.)

- Take x_{41} and x_{23} time-like:



- Certainly looks like a 'scattering amplitude'
- Claim:

$$S \equiv \frac{G}{G_{\mathrm{Eucl}}} \quad \text{satisfies} \quad |S| \leq 1$$

proof

- s-channel OPE diverges upon entering light-cone

- Use OPE around t-channel (timelike one)

$$G(\rho,\bar\rho)=\sum_{j,\Delta} c_{j,\Delta}\left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^{\Delta-j}\left(\frac{1-\sqrt{\bar\rho}}{1+\sqrt{\bar\rho}}\right)^{\Delta+j}$$

[Hogervorst&Rychkov '13]

- For timelike, $\rho>1$, only get extra phases:

$$\begin{aligned}\big|G(\rho,\bar\rho)\big| &= \Big|\sum(\text{positive})e^{i\pi(\Delta-j)}\Big| \\ &\le \sum(\text{positive}) = G(1/\rho,\bar\rho)\equiv G_{\text{Eucl}}\end{aligned}$$

This means that an ‘imaginary part’ is positive:

$$\begin{matrix} S = 1 + i\mathcal{M} \\ |S| \leq 1 \end{matrix} \quad \Rightarrow \mathrm{Im}\,\mathcal{M} > 0$$

Since S contains the ‘1’, this is *double discontinuity*:

$$\begin{aligned} G_{\mathrm{Eucl}} &\propto 1 \\ G_{\mathrm{below}} &\propto 1 + i\mathcal{M} \\ G_{\mathrm{above}} &\propto 1 - i\mathcal{M}^* \end{aligned}$$

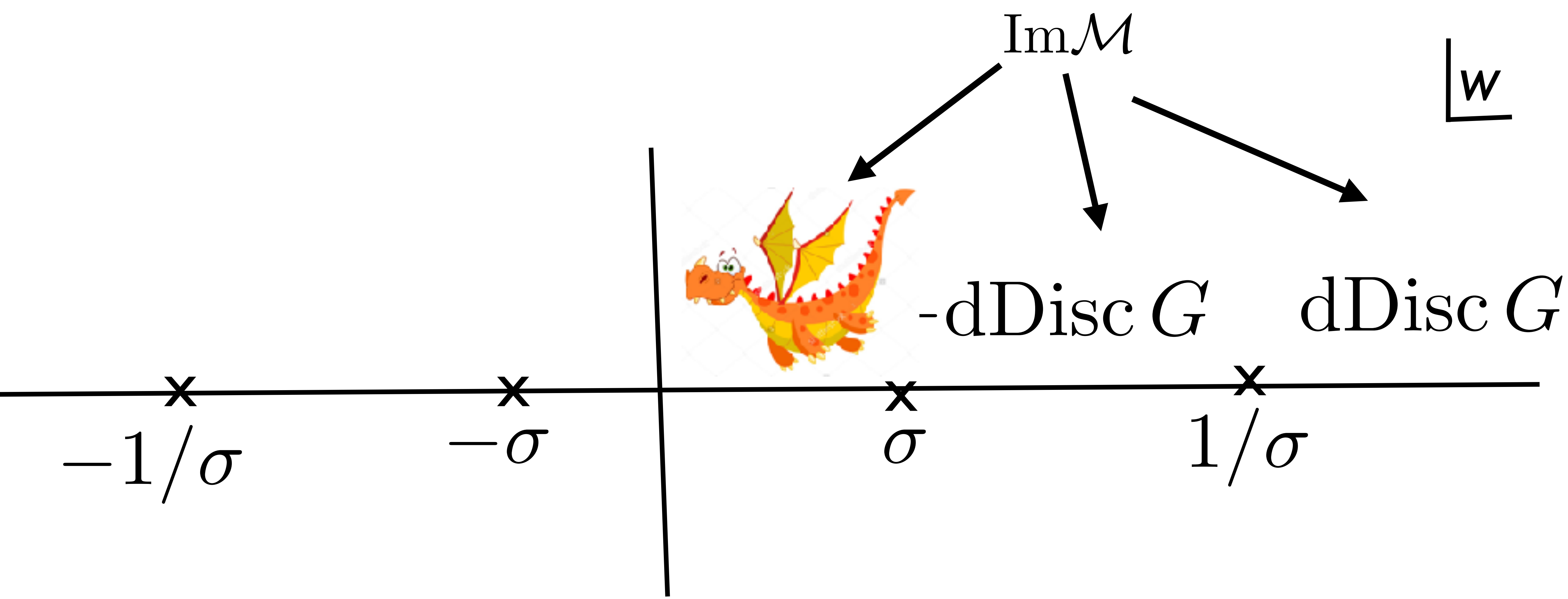
$$\begin{aligned} \Rightarrow 2\mathrm{Im}\,\mathcal{M} &\propto 2G_{\mathrm{Eucl}} - G_{\mathrm{above}} - G_{\mathrm{below}} \\ &\equiv \mathrm{dDisc}\,G \\ &> 0 \end{aligned}$$

Writing M from Im M: 'dispersion relation'

This doesn't quite work because analytic structure in coordinate space is weird

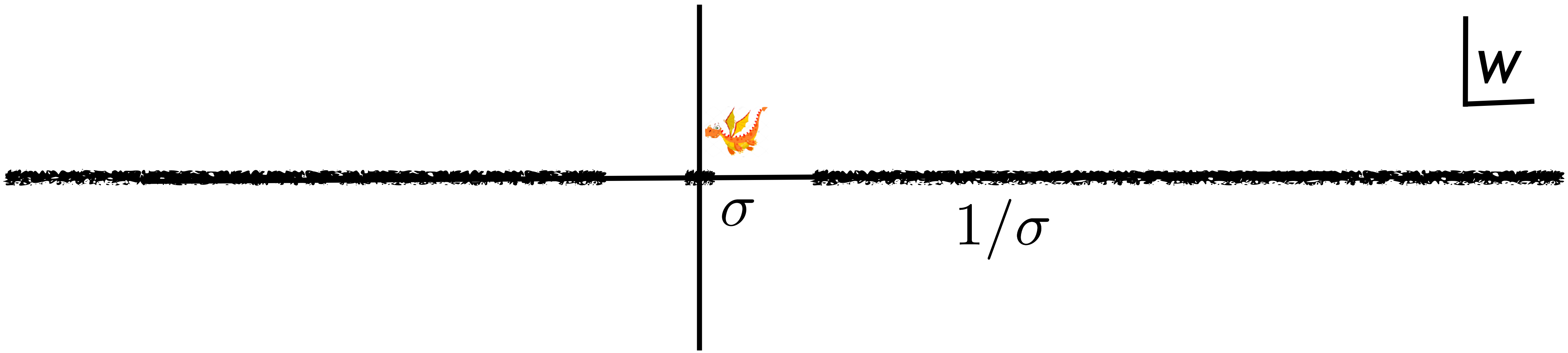
For example, fix $\rho\bar{\rho}$: $\rho = \sigma w$

$\bar{\rho} = \sigma / w$



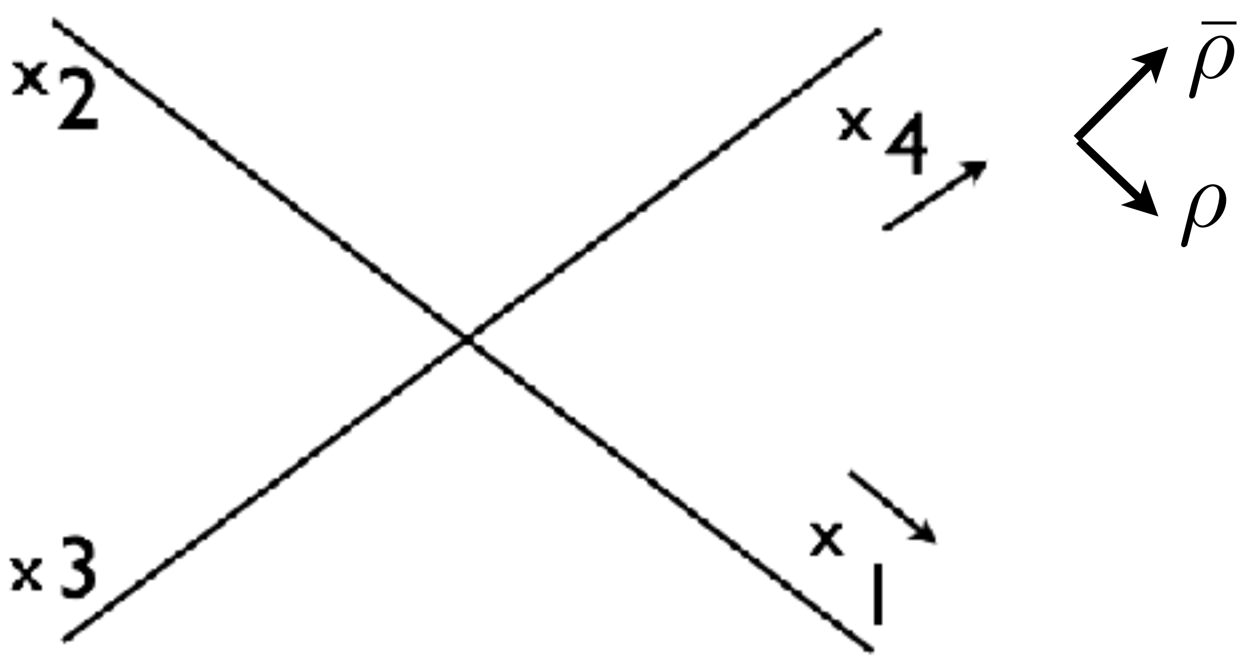
The dragons shrink in the Regge limit

$$w \to \infty$$

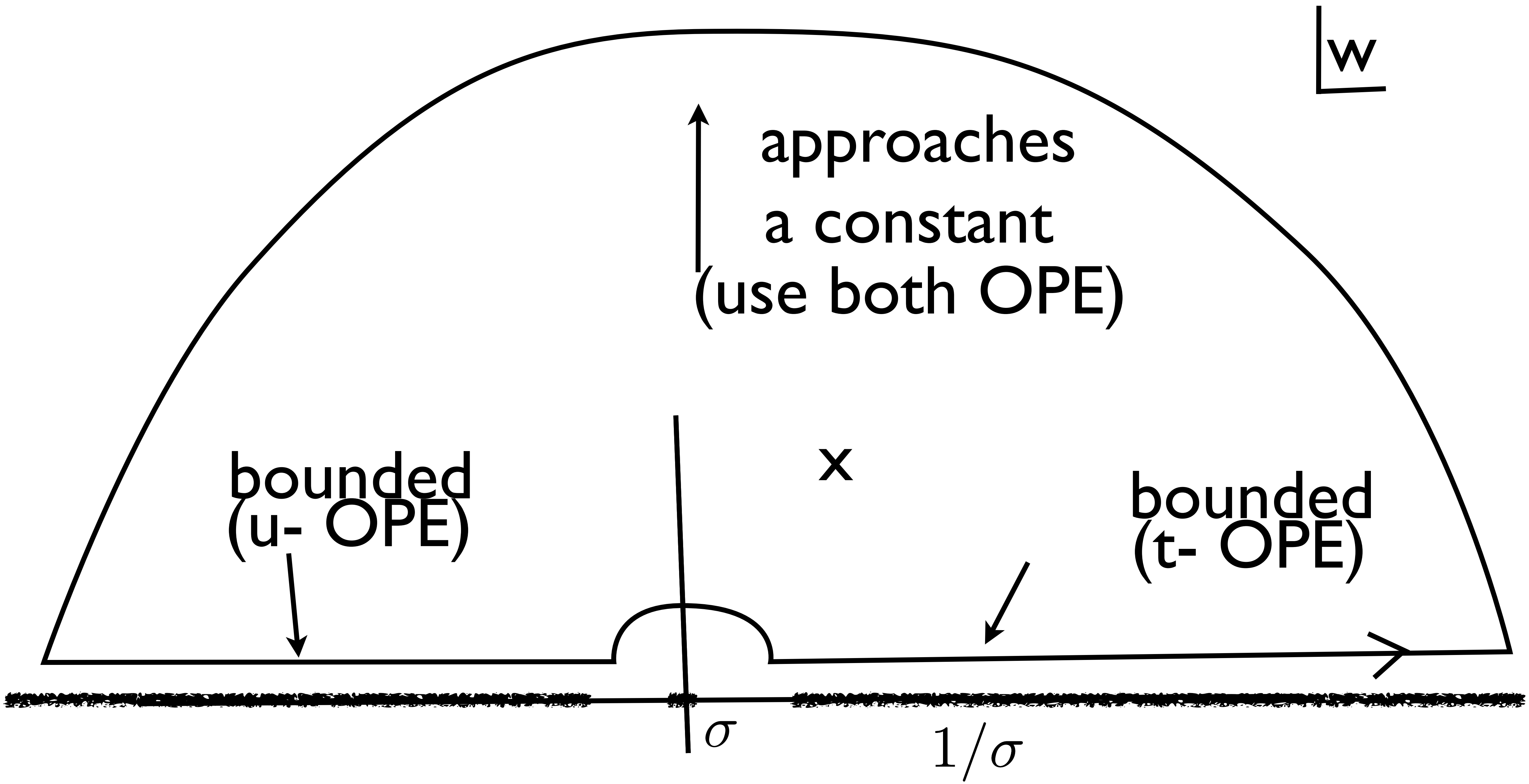


This corresponds to a boosted coordinates:

$$\rho = \sigma w$$
$$\bar{\rho} = \sigma / w$$



Regge limit dispersion relation:



$$\mathcal{M}(E) = C + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dE' \,\mathrm{Im}\,\mathcal{M}(E')}{E - E'}$$

[Hartman, Kundu & Tajdini '16]

Some implications of the dispersion relation:

Look in upper-half-plane:

$$\mathrm{Im}\,\mathcal{M}(x+iy) = \int \frac{y\,dx'\,\mathrm{Im}\mathcal{M}(x')}{(x'-x)^2+y^2} > 0$$

⇒Proof of ANEC:

$$\mathcal{M}(w) \approx w\langle \int_{-\infty}^{\infty} dx^+ T_{++}\rangle_{34} \;\Rightarrow \langle \int dx^+ T_{++}\rangle > 0$$

[Hartman,Kundu&Tajdini '16]

(ANEC was proved just a few month earlier using entanglement entropy inequalities)

[Faulkner,Leigh,Parrikar&Wang, '16]

Derivative of dispersion relation:

$$(y\partial_y - 1)\Big[\log \mathrm{Im}\mathcal{M}(x+iy)\Big] = -2\frac{\int \frac{dx'\, y^2\, \mathrm{Im}\mathcal{M}(x')}{((x'-x)^2+y^2)^2}}{\int \frac{dx'\, \mathrm{Im}\mathcal{M}(x')}{(x'-x)^2+y^2}} \leq 0$$

$\Rightarrow$ can't grow faster than linear in energy!

This proves that the Pomeron intercept $j \leq 2$ in CFT

when converted to Rindler time $w=e^{t/(2\pi T)}$, this is the CFT case of the 'bound on chaos' (Lyapunov $\lambda < 2\pi T$)

[Maldacena,Shenker&Stanford '15]

$\Rightarrow$ Dispersion relation encodes much nice physics!

Because of the ‘dragons’ at low-energy, this dispersion relation doesn’t fully reconstruct the correlator

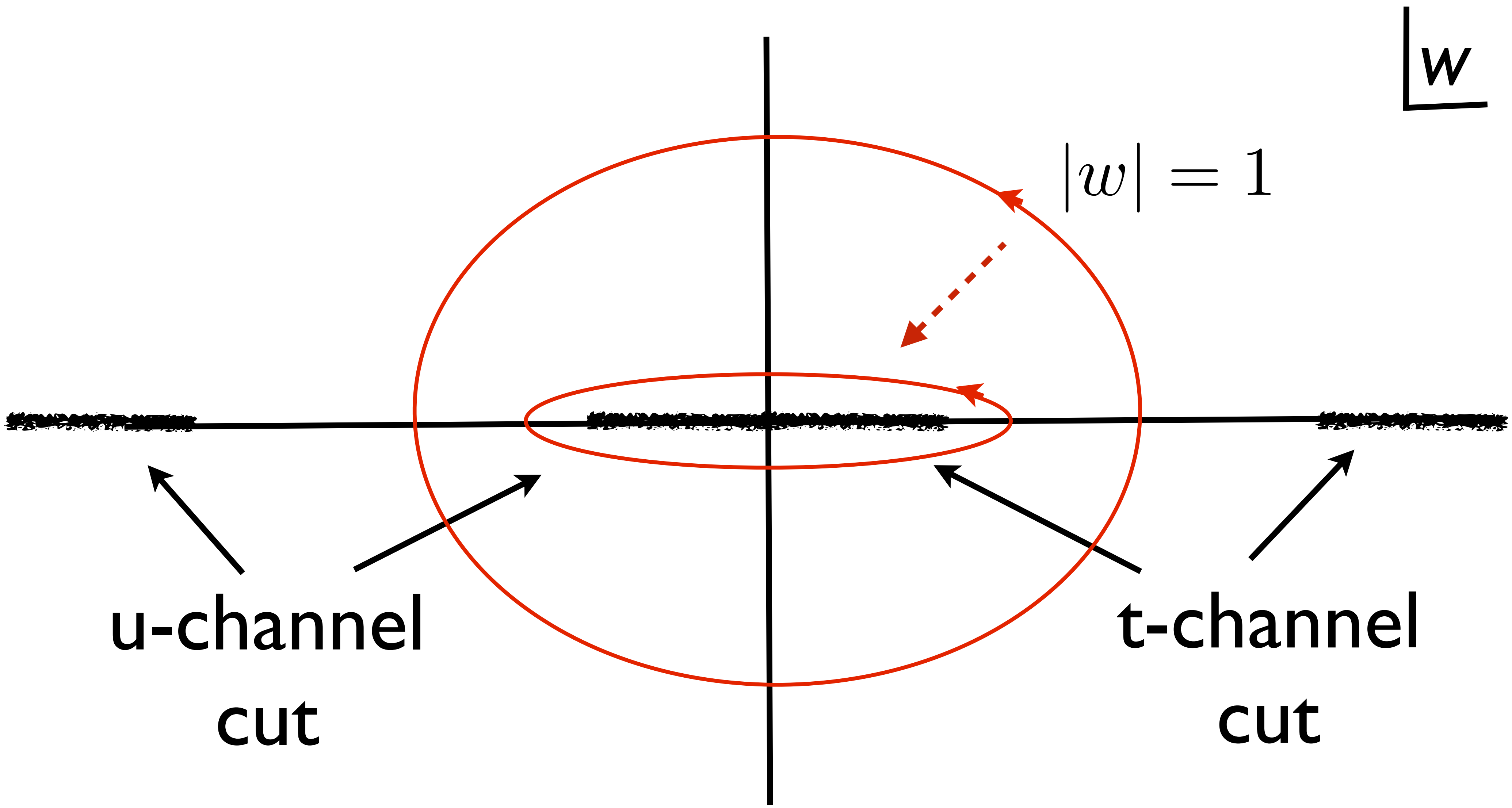
Following the Froissart-Gribov logic, we’ll instead obtain a dispersion relation for OPE coefficients

Froissart-Gribov: how to invert $f(\cos\theta) = \sum_{j=0}^{\infty} f_j \cos(j\theta)$?

A: Start from Euclidean inverse, use variable: $w = e^{i\theta}$

$$f_j \sim \int_0^{2\pi} d\theta \cos(j\theta) f(\cos\theta)$$
$$= \oint \frac{dw}{w}(w^j + w^{-j}) f(\cos\theta)$$

Trick is to close the contour on cut:



Result: integral over cut

$$f_j^{(t)} \sim \int_{\eta_0}^{\infty} d\eta \, e^{-j\eta} \, \mathrm{Disc}\, f(\cosh(\eta))$$

$$+(-1)^j [\text{u-channel}]$$

CFT steps are the same: Euclidean OPE:

$$G(z,\bar{z}) = \sum_{j,\Delta} f^2_{j,\Delta} G_{j,\Delta}(z,\bar{z})$$

- Actually, we first have to make Δ continuous:

$$G(z,\bar{z}) = \delta_{12}\delta_{34} + \sum_{j=0}^{\infty} \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i}\, c(j,\Delta)\, F_{j,\Delta}(z,\bar{z}).$$

✓

$$F_{j,\Delta} = g_{j,\Delta} + g_{j,d-\Delta}$$

[Costa,Goncalves&Penedones'12]

[see also: Mazac'16

Hogervorst&van Rees '17, Gadde '17

= single-valued, needed for self-adjointness of Casimir

[Simmons-Duffin '12]

- Contour like a Mellin transform

- OPE reproduced if c has correct poles:

$$c(j,\Delta') \approx \frac{f^2_{OO\to j,\Delta}}{\Delta - \Delta'}$$

step 1. Invert the Euclidean OPE (SO(d+1,1)):

- To extract the coefficients, invoke orthogonality and integrate against block (+shadow):

$$c(j,\Delta) = \#(j,\Delta) \int d^2z\, \mu(z,\bar{z})\, G(z,\bar{z})\, F_{j,\Delta}(z,\bar{z})\,.$$

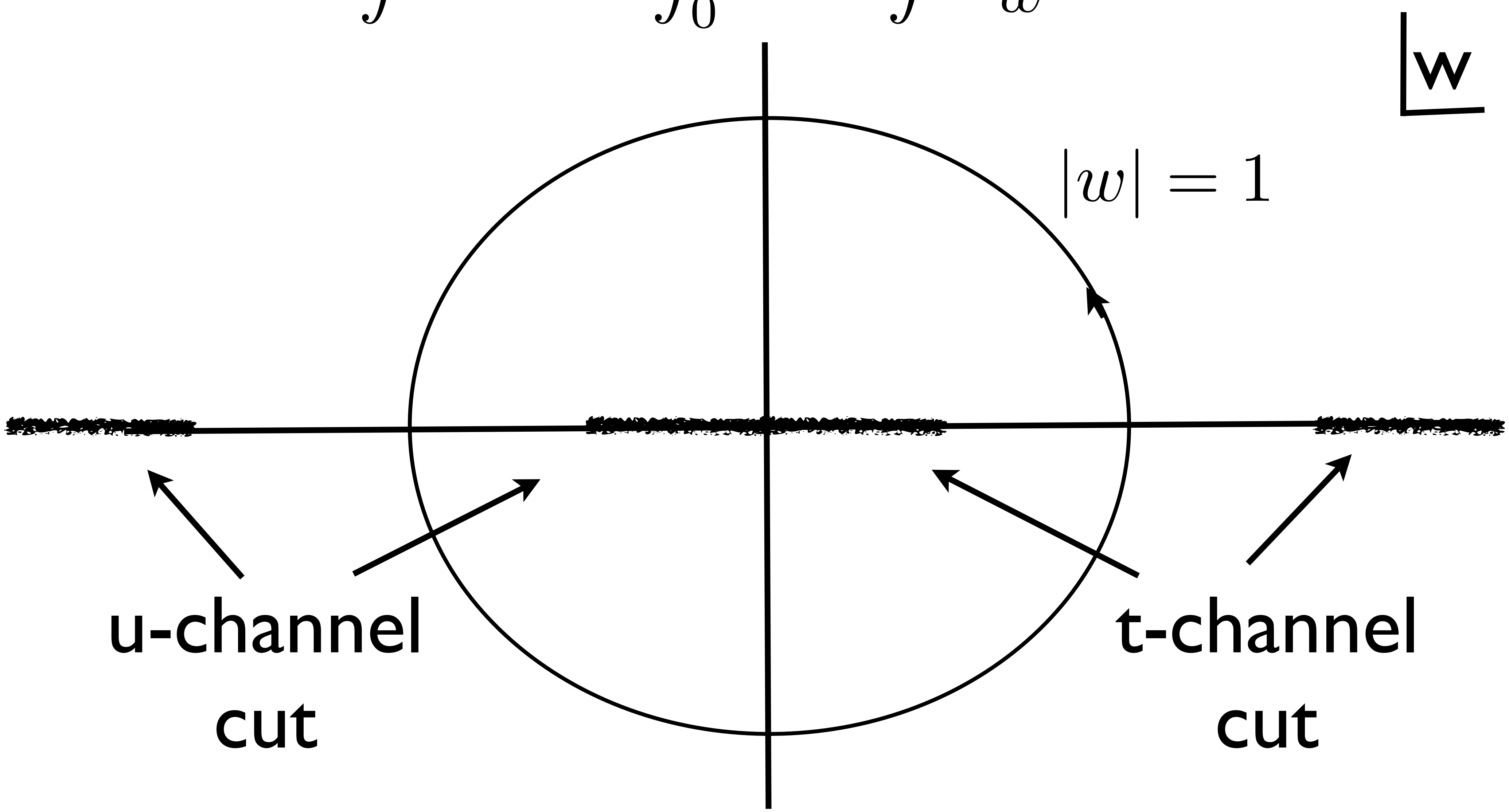
- Still a Euclidean integral: integer spin, not yet what we want step 2

- Contour deformation. Use clever variables

[Hogervorst&Rychkov '13]

$$z = \frac{4\rho}{(1+\rho)^2} \qquad w = \sqrt{\rho/\bar{\rho}} = e^{i\theta}$$

$$\int d^2 z \to \int_0^1 d|\rho| \oint \frac{dw}{w}$$



The tricky part is to split the block(+shadow) into bits that are nice in individual Regge limits:

$$2\cos(j\theta) = e^{ij\theta} + e^{-ij\theta}$$
$$= w^{j} + w^{-j}$$

That is, we want:

$$F_{j,\Delta}(z,\bar{z}) = F^{(+)}_{j,\Delta} + F^{(-)}_{j,\Delta}$$

$\sim w^{j}$

$(w \to 0)$

$\sim w^{-j}$

$(w \to \infty)$

- tricky because there are 8 basic solutions to conformal Casimirs diff eqs.: (quadratic and quartic)

$$g^{\text{pure}}_{j,\Delta}(z,\bar{z}) \sim z^{\frac{\Delta-j}{2}}\,\bar{z}^{\frac{\Delta+j}{2}} \qquad (0 \ll z \ll \bar{z} \ll 1)$$

- Solutions related by symmetries:

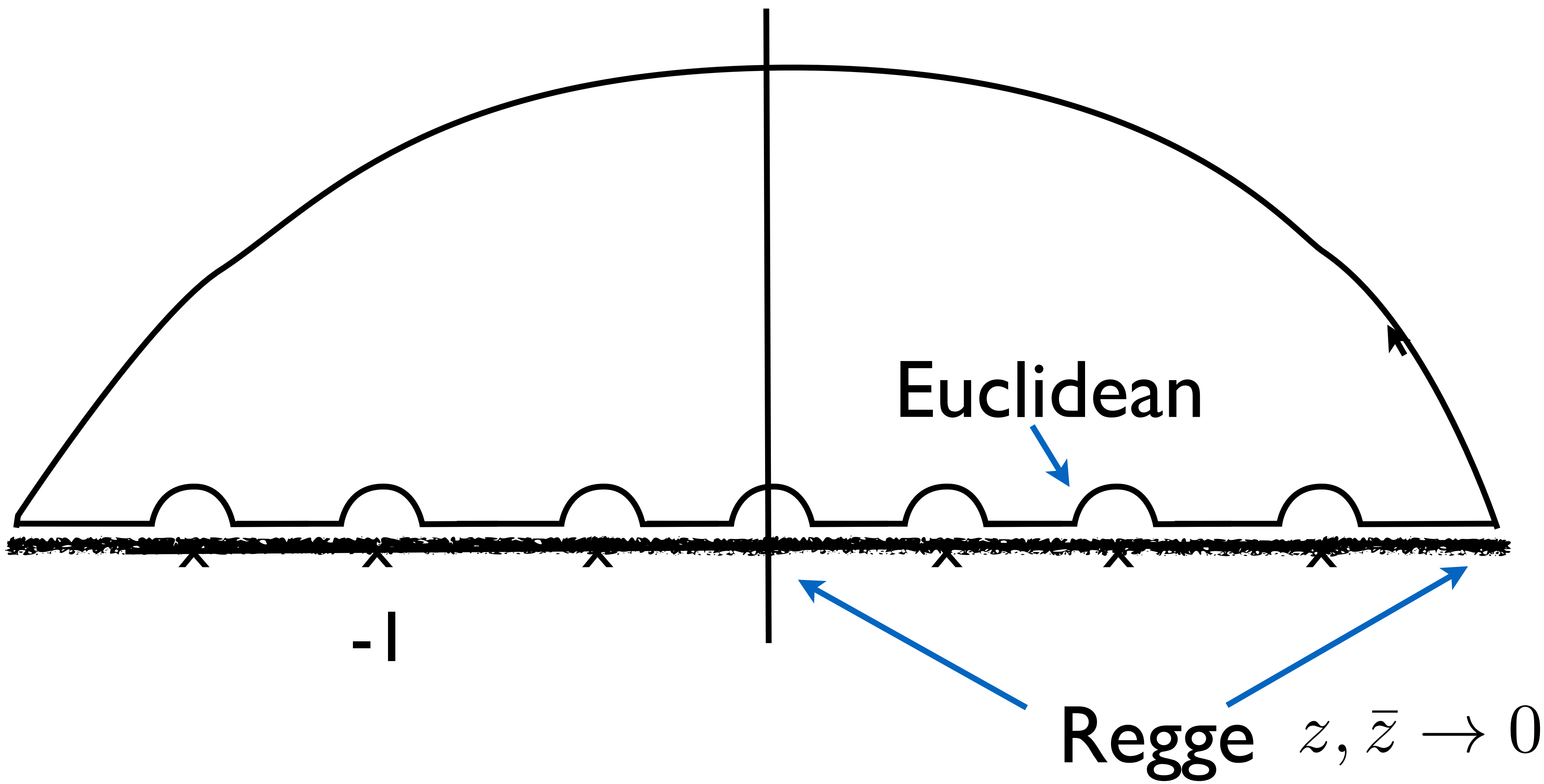
$$j \longleftrightarrow 2-d-j, \qquad \Delta \longleftrightarrow d-\Delta, \qquad \Delta \longleftrightarrow 1-j\,.$$

- Only 2 are nice (convergent) in Regge limit:

$$g^{\text{pure}}_{\Delta+1-d,j+d-1}, \quad g^{\text{pure}}_{1-\Delta,j+d-1} \sim (z\bar{z})^{j/2}$$

- So we have 4 parameters and 8 constraints

- The constraints are on different sheets:



(all 8 solutions mix under continuation:)

$$g^{\text{pure}}_{j,\Delta}(z,\bar{z})^{\circlearrowleft} = g^{\text{pure}}_{j,\Delta}(z,\bar{z})\left[1 - 2i\frac{e^{-i\pi(a+b)}}{\sin(\pi\beta)}\sin(\pi(\beta/2+a))\sin(\pi(\beta/2+b))\right] - g^{\text{pure}}_{1-\Delta,1-j}(z,\bar{z})\,2\pi i\frac{e^{-i\pi(a+b)}\Gamma(\Delta+j-1)\Gamma(\Delta+j)}{\Gamma\left(\frac{\Delta+j}{2}-a\right)\Gamma\left(\frac{\Delta+j}{2}+a\right)\Gamma\left(\frac{\Delta+j}{2}-b\right)\Gamma\left(\frac{\Delta+j}{2}+b\right)}\,.$$

4 parameters, 8 constraints,
fingers crossed…

Result: CFT Froissart-Gribov formula

$$c(J,\Delta) = c^t(J,\Delta) + (-1)^J c^u(J,\Delta)$$

$$\boxed{\begin{aligned} &c^t(J,\Delta) = \\ &\frac{\kappa_{J+\Delta}}{4}\int_0^1 dz d\bar{z}\, \mu(z,\bar{z})\, G_{\Delta+1-d,J+d-1}(z,\bar{z})\, \mathrm{dDisc}\left[G(z,\bar{z})\right]\end{aligned}}$$

Result: CFT Froissart-Gribov formula

$$c(J,\Delta) = \int_{\Diamond} \left[\text{Inverse block}\right] \times \left[\text{dDisc}\, G\right]$$

s-channel
OPE coefficients

convergent
t-channel sum

Result: CFT Froissart-Gribov formula

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s-channel
OPE coefficients

block with
j and Δ
exchanged

convergent
t-channel sum

converges for j>1 (Regge limit bounds)

A (boring) test: 2D Ising

$$G(\rho,\bar\rho) = \left|\frac{1}{(1-\rho^2)^{1/4}}\right|^2 + \left|\frac{\sqrt{\rho}}{(1-\rho^2)^{1/4}}\right|^2$$

- Double discontinuity:

$$\frac{1-\frac{1}{\sqrt{2}}(\sqrt{\rho}+\sqrt{\bar\rho})+\sqrt{\rho\bar\rho}}{(1-\rho^2)^{1/4}(1-\bar\rho^2)^{1/4}} > 0$$

- Integral against 2d (global) blocks: factorize

$$c_{j,\Delta} = f_0(j{+}\Delta)f_0(j{+}2{-}\Delta) - \tfrac{1}{2}f_{1/4}(j+\Delta)f_0(j+2-\Delta) + \ldots$$

$$f_p(\alpha) = 2^{a-3+2p}\frac{\Gamma(\frac{7}{4})\Gamma(p+\frac{\alpha-2}{4})}{\Gamma(p+\frac{\alpha+5}{4})}\,{}_3F_2\big(\tfrac{1}{2},\tfrac{\alpha}{2},p+\tfrac{\alpha-2}{4};\tfrac{a+1}{2},p+\tfrac{\alpha+5}{4};1\big)\,. \qquad \text{(B.6)}$$

- Residues at all poles do match global OPE!*

$$C_{j,\Delta} = -K_{j,\Delta}\mathrm{Res}_{\Delta'=\Delta}c(j,\Delta')$$

$$C_{0,1} = \frac{1}{4}, \qquad C_{2,2} = \frac{1}{64}, \qquad C_{4,4} = \frac{9}{40960}, \qquad C_{0,4} = \frac{1}{4096}$$
$$C_{4,5} = \frac{1}{65536}, \qquad C_{6,6} = \frac{35}{3670016} \qquad C_{2,6} = \frac{9}{2621440}, \qquad C_{6,7} = \frac{1}{1310720}, \ldots$$

* Including (predicted) spurious poles for $\Delta - j - d = 0, 1, 2 \ldots$

$$\frac{(-1)^{m+1}\Gamma(1+a+\frac{m}{2})\Gamma(1+b+\frac{m}{2})}{m!(m+1)!\Gamma(a-\frac{m}{2})\Gamma(b-\frac{m}{2})}\times$$
$$\times K_{j+1+m,j+d-1}c(j+1+m, j+d-1)$$

**And: never trust Mathematica's Residue on 3F2........

Outline

The aim of this talk will be to present a formula…

1. Context: the conformal bootstrap ✔
2. An inverse OPE formula:
   -Why operators are analytic in spin
   -building it up: SO(2),SO(3),SO(2,1)…SO(d,2) ✔
3. Applications:
   -operators of large spin
   -CFTs dual to gravity: causality&bulk locality

Large spin operators

- Physical intuition: take two operators and put many derivatives between them:

$$\mathcal{O}_{\#} = \mathcal{O}_1 \partial^{\#} \mathcal{O}_2$$

- Should make them 'far' and decoupled:

$$\Delta_{\#} \approx \Delta_1 + \Delta_2 + (\# \text{ derivatives})$$ (as in free theory!)

- Actually, twist is more useful: $\tau = \Delta - j$

$$\tau_{[12]_n} \approx \tau_1 + \tau_2 + 2n$$

Large spin expansions

large spin in s-channel $\leftarrow$ low twist in t-channel

standard story: double-light-cone limit $(z, \bar{z}) \to (0, 1)$

non-analytic behaviour in $(1-\bar{z})$ needs large spin:

$$\sum_j \frac{1}{j^\alpha} F_j(\bar{z}) = (1-\bar{z})^{\alpha/2} + \text{regular}$$

$\Rightarrow$ Solve OPE in asymptotic series in 1/j

[Komargodski&Zhiboedov,
Fitzpatrick,Kaplan,Poland&Simmons-Duffin,
Alday&Bissi&...,
,Kaviraj,Sen,Sinha&...,
Alday,Bissi,Perlmutter&Aharony,...]

- New: start instead from Froissart-Gribov formula:

$$c(j,\Delta) \sim \int_0^1 dz d\bar{z}\, z^{j-\Delta}\, \bar{z}^{j+\Delta} F_{j+\Delta}(\bar{z}) \mathrm{dDisc} G(z,\bar{z})$$

- Recall, OPE data encoded in Δ-poles: $z \to 0$

if $G(z,\bar{z}) \to z^{\tau} G_{\tau}(\bar{z})$,

$$c(j,\Delta) = \frac{1}{j-\Delta-\tau} \times \int_0^1 d\bar{z}\, \bar{z}^{j+\Delta} F_{j+\Delta}(\bar{z}) \mathrm{dDisc} G_{\tau}(\bar{z})$$

- Large j+Δ and low twist pushes to (0,1) corner

- Analytic result for collinear integral of power:

$$I^{a,b}_{\tau'}(\bar h) \equiv \int_0^1 \frac{d\bar z}{\bar z^2}(1-\bar z)^{a+b}\kappa_{\bar h} k_{\bar h}(\bar z)\,\mathrm{dDisc}\left[\left(\frac{1-\bar z}{\bar z}\right)^{\frac{\tau'}{2}-b}(\bar z)^{-b}\right] \quad (4.7)$$

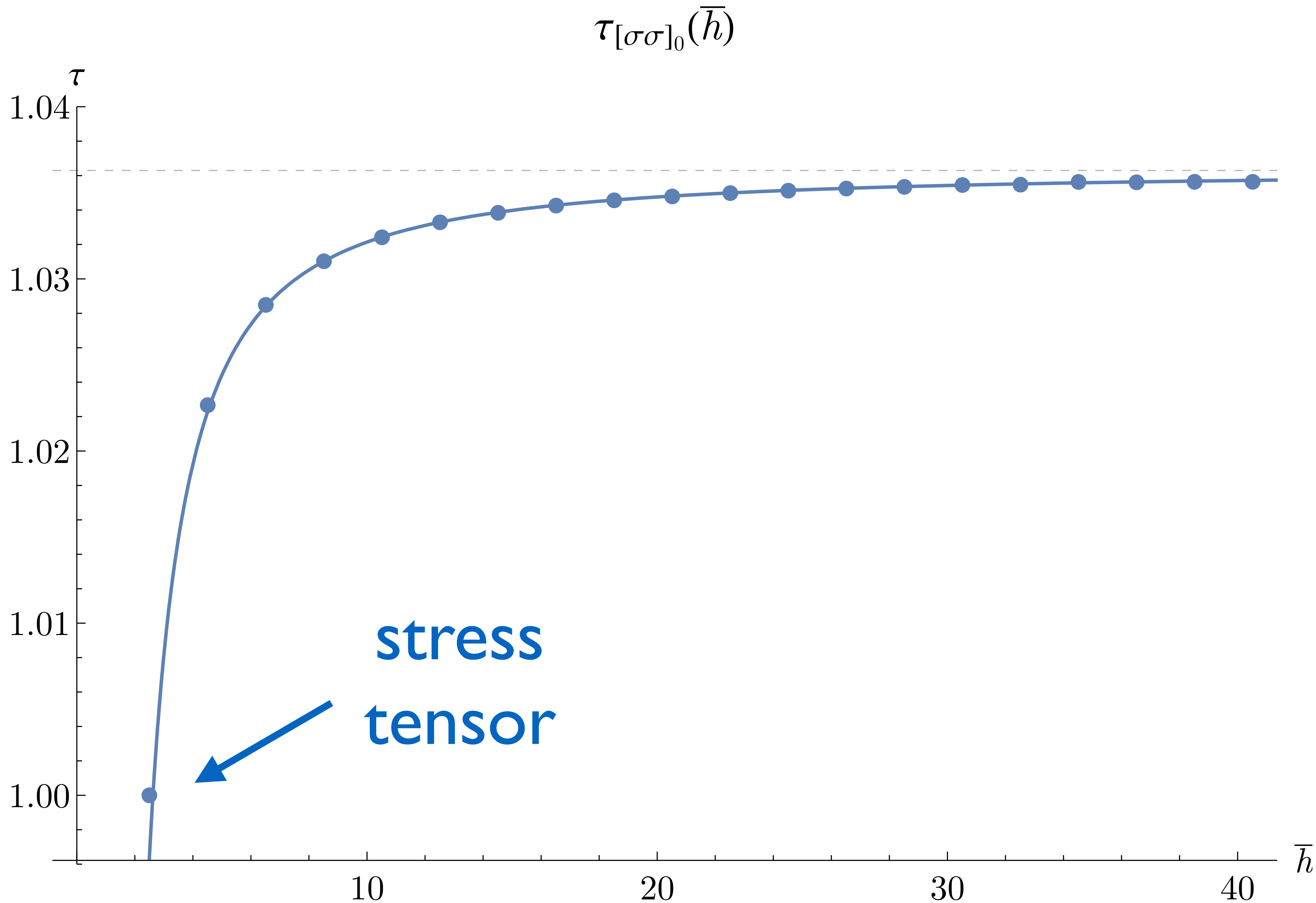
$$= \frac{1}{\Gamma\big(-\frac{\tau'}{2}-a\big)\Gamma\big(-\frac{\tau'}{2}+b\big)} \times \frac{\Gamma\big(\bar h - a\big)\Gamma\big(\bar h + b\big)}{\Gamma\big(2\bar h - 1\big)} \times \frac{\Gamma\big(\bar h - \frac{\tau'}{2} - 1\big)}{\Gamma\big(\bar h + \frac{\tau'}{2} + 1\big)}\,.$$

$$\sim 1/\bar h^{\tau'} \qquad (\bar h = \tfrac{j+\Delta}{2})$$

- Earlier results reproduced by: ‘expand cross-channel OPE in $\frac{1-\bar z}{\bar z}$ and integrate termwise using (4.7)’

- Conceptually, no need to expand in 1/j
  (=why earlier expansions were asymptotic) [Alday&Zhiboedov ’15; Simmons-Duffin ’16]

Asymptotic series in 3D Ising



$$\bar{h} = \frac{j+\Delta}{2}$$

[Plot from Simmons-Duffin '16;
see Alday&Zhiboedov '15]

What's new:

- Asymptotic expansion ⇒ convergent sum

  (no need to expand in $(1-\bar{z})/\bar{z}$)
- Control over individual spins, not only averages over many spins ('no stick-out')
- Can try to bound errors

AdS/CFT: Why dDisc is awesome

In theories with large-N factorization, saturated by single-traces

$$G_{\text{conn}} \sim \sum_{\text{single trace}} c_j |1-z|^{\Delta_j} + \sum_{\text{double trace}} c_i |1-z|^{\gamma_i^{(1)}/N}$$

Leading connected order $\sim \log(1-\bar{z})$

$$c_{j,\Delta} \sim \int \mathrm{dDisc}\, G$$

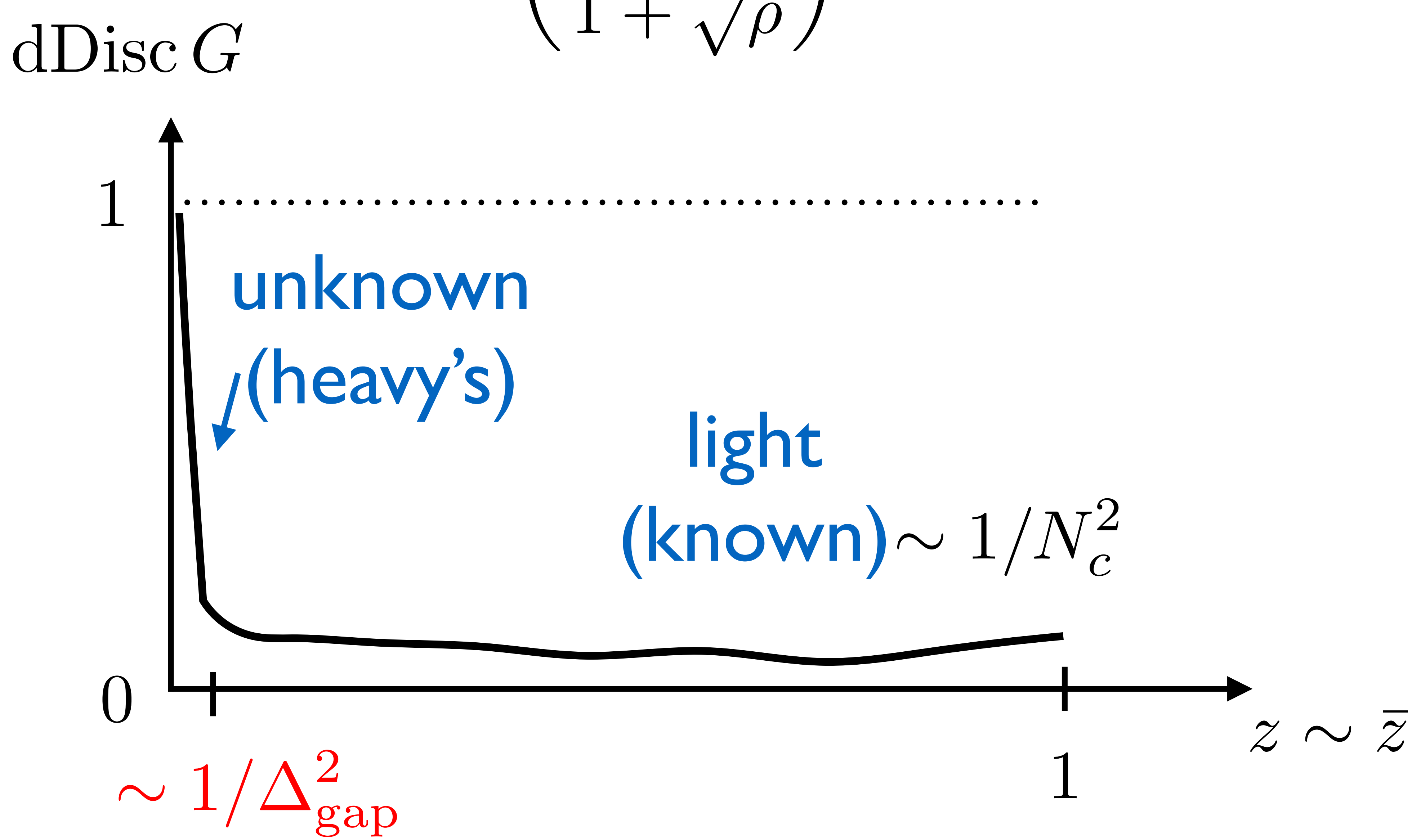
single+double traces

single trace

Theories with gravity dual have few light single-traces: the graviton ($T^{\mu\nu}$), a few scalars,… up to Δ_{gap}

Heavy's in cross-channel can be bounded:

$$\left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^{\Delta_{\rm gap}} \leq e^{-2\sqrt{\rho}\Delta_{\rm gap}}$$



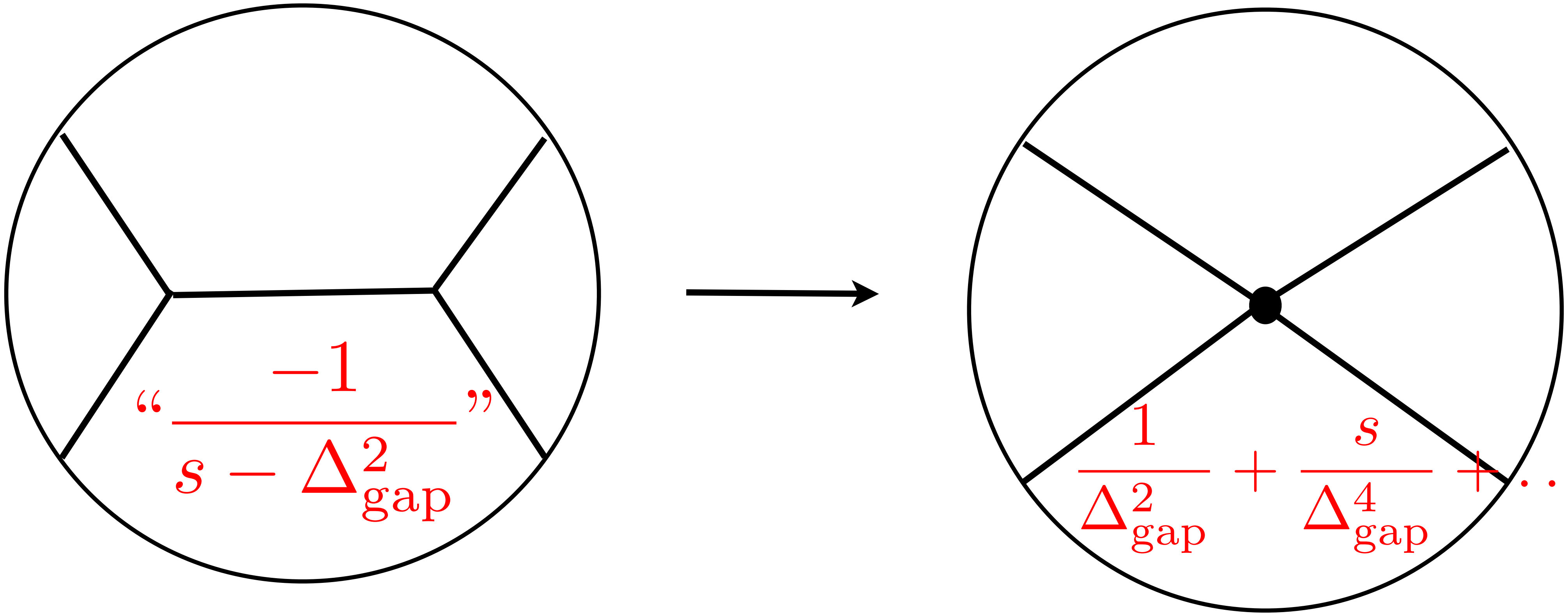
s-channel OPE coefficients ~ weighted areas

$$c(j, \tfrac{d}{2} + i\nu) \sim \int_0^1 d(z\bar{z})(z\bar{z})^{j/2-1} e^{-(\sqrt{z}+\sqrt{\bar{z}})\Delta_{\text{gap}}} \times \int d(z/\bar{z})(\ldots)$$

$$\propto \frac{1}{(\Delta_{\text{gap}}^2)^{j-1}}$$

Area itself is (inverse) stress-tensor two-point function

$$\boxed{\Rightarrow \quad \left|c(j, \tfrac{d}{2} + i\nu)_{\text{heavy}}\right| \leq \frac{1}{c_T} \frac{\#}{(\Delta_{\text{gap}}^2)^{j-2}}}$$



- AdS/CFT expectation: EFT coefficients suppressed by dimension: $(\partial^{2k})\phi^4$ down by $1/\Delta_{\text{gap}}^{2k}$

- What this shows: down by spin $1/\Delta_{\text{gap}}^{2(j-2)}$

- Same as ‘causality bound’ conjectured recently (equivalent to existence of dispersion relation in the bulk)
[Maldacena,Simmons-Duffin&Zhiboedov ’15]

- What was known from CFT:
  Solutions to crossing symmetry in a large-N CFT with large gap = Witten diagrams

  [Heemskerk,Penedones, Polchinski& Sully '09]

  For given light spectrum, solutions are ambiguous by contact interactions

- What we learn:
  analyticity in spin (good Regge behavior) singles out a unique, 'causal' solution, up to errors:

  $$< 1/(\Delta_{\text{gap}}^2)^{j-2}$$

- key step toward AdS/CFT from CFT!

(Spin versus dimension)

- Some sporadic few-derivative interactions remain unconstrained
- Consider an AdS interaction with flat-space limit:

$$stu$$

- This has spin two in the Regge limit in all channels:

$$stu = st(s+t) \sim s^2 \equiv s^j \qquad (s \to \infty, t \text{ fixed})$$

- All interactions with more derivative, however, must have small coefficients

Conclusion

- Novel formula: *(Froissart-Gribov-like)*

$$c(j,\Delta) \equiv \int_0^1 dz d\bar{z}\, g_{\Delta,j}\, \mathrm{dDisc}\, G$$

s-channel t-channel

- Valid in any unitary $\mathrm{CFT_D}$. Regge behavior ensures convergence; ounds derivative interactions in AdS

- Opens the way for AdS/CFT beyond gravity

- Outlook:
  -Numerical bootstrap: bound errors using convergent 1/j expansion?
  -Higher points?

& much more!

Interpretation of Q_j

- Legendre polys P_j: finite dim representations, with J_z = -j…j:

Ex: $$P_4(\cos\theta) \propto 35e^{4i\theta} + 20e^{2i\theta} + 18 + 20e^{-2i\theta} + 35e^{-4i\theta}$$

- Q_j functions associated with infinite-dim highest-weight reps:

$$Q_j(\cosh\eta) = e^{-(j+1)\eta} + \#e^{-(j+3)\eta} + \#'e^{-(j+5)\eta} + \dots$$

- Q_j = second solution to Legendre diff eq.