Small x Asymptotics of the Quark Helicity Distribution

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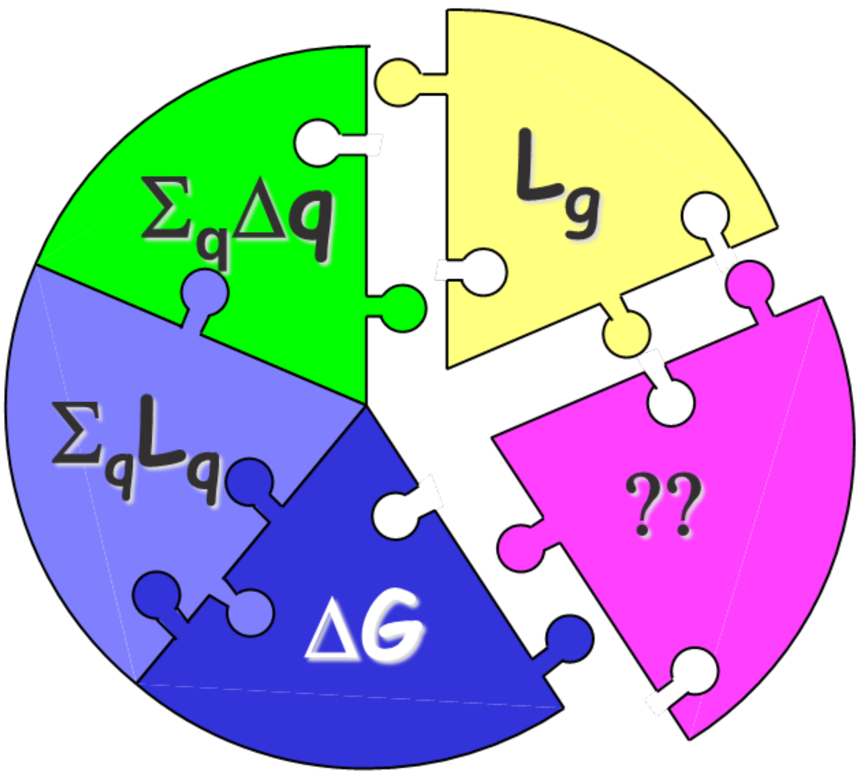
arXiv:1610.06188 [hep-ph]
arXiv:1703.05809 [hep-ph]

Outline

- Goal: understanding the proton spin coming from small x
- Small-x helicity evolution:
  - Observables: quark helicity TMD & PDF at small-x, g_1 structure function
  - Small-x evolution for the "polarized dipole": large-N_C limit
- Numerical solution of the large-N_C evolution equations:
  - small-x asymptotics of the g_1 structure function, quark hPDFs and helicity TMDs
  - impact on proton spin
- Analytic solution of the large-N_C evolution equations:
  - Pure joy

Our Goal: Proton Spin at Small x

Proton Spin Pie Chart



- The proton spin carried by the quarks is estimated to be (for $0.001 < x < 1$)

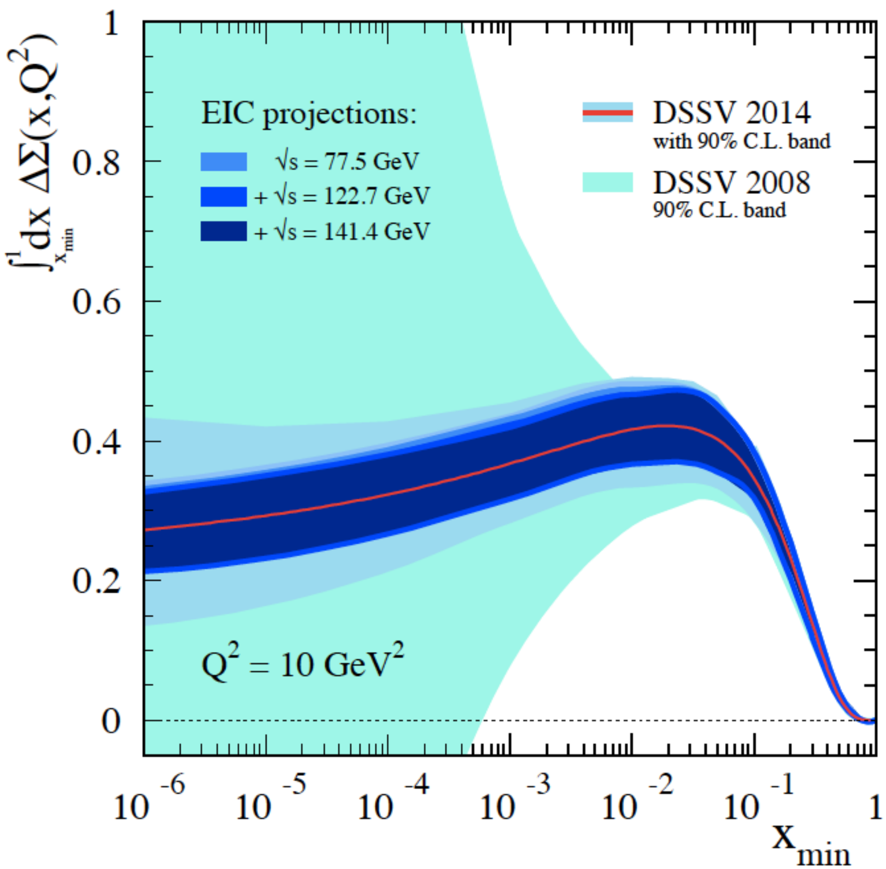
$$S_q(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.15 \div 0.20$$

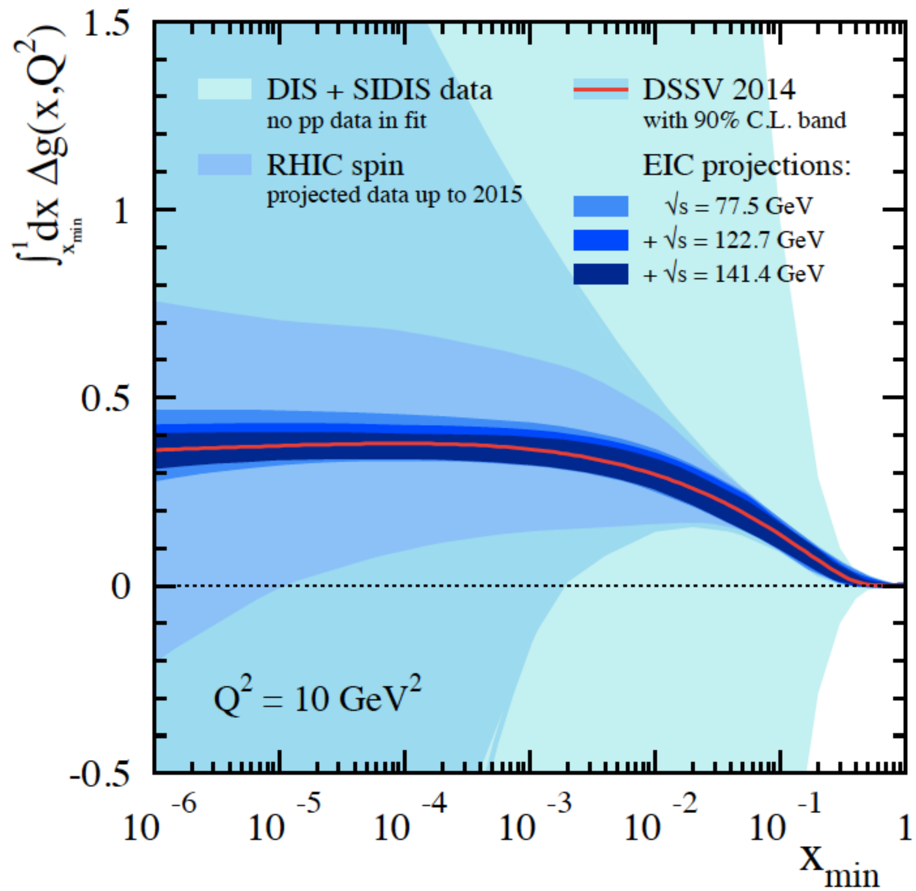
- The proton spin carried by the gluons is (for $0.05 < x < 1$)

$$S_G(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.13 \div 0.26$$

- Unfortunately the uncertainties are large. Note also that the x-ranges are limited, with more spin (positive or negative) possible at small x.

How much spin is at small x?





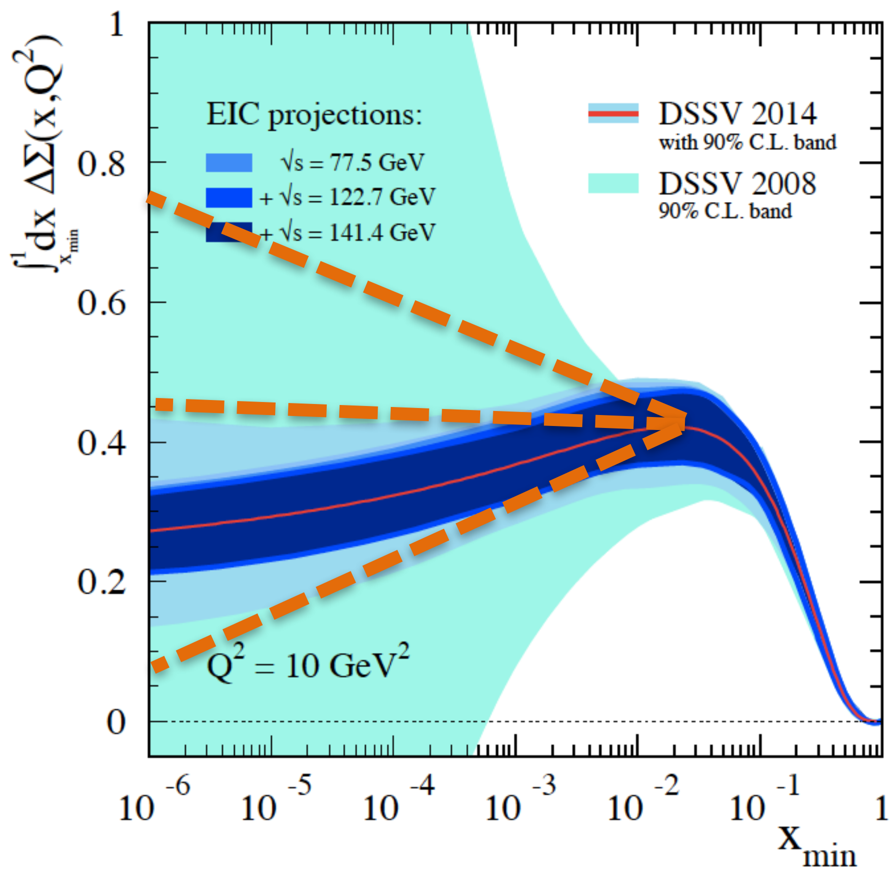
- E. Aschenaur et al, arXiv:1509.06489 **[hep-ph]**
- Uncertainties are very large at small x!

Spin at small x

- The goal of this project is to provide theoretical understanding of helicity PDF's at very small x.

- Our work would provide guidance for future hPDF's parametrizations of the existing and new data (e.g., the data to be collected at EIC).

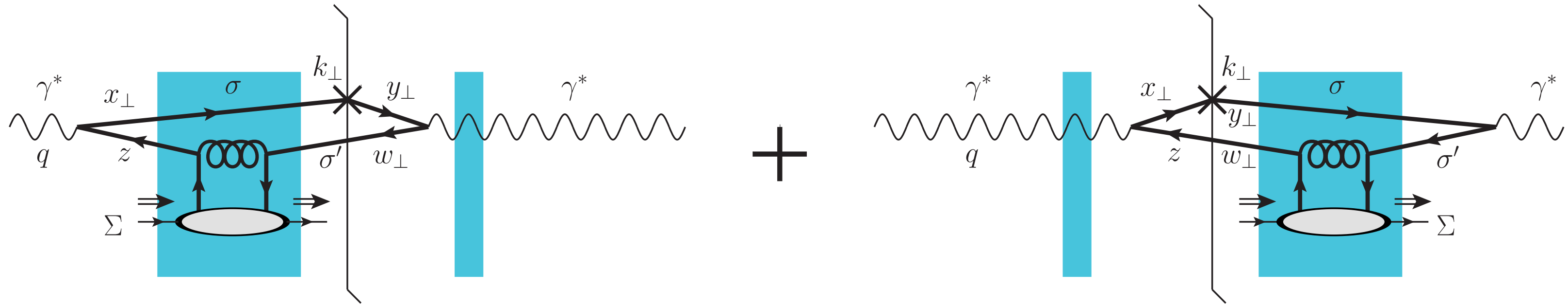
- Strictly-speaking we only talk about quark helicity, but most likely our analysis applies to gluon hPDF's as well.



Helicity Evolution Equations at Small x
flavor-singlet case

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph],
arXiv:1610.06197 [hep-ph]

Quark Helicity Observables at Small x



- One can show that the g_1 structure function and quark helicity PDF (Δq) and TMD at small-x can be expressed in terms of the polarized dipole amplitude (flavor singlet case):

$$g_1^S(x,Q^2) = \frac{N_c\,N_f}{2\,\pi^2 \alpha_{EM}} \int\limits_{z_i}^{1} \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2} \sum_{\lambda\sigma\sigma'} |\psi^T_{\lambda\sigma\sigma'}|^2_{(x_{01}^2,z)} + \sum_{\sigma\sigma'} |\psi^L_{\sigma\sigma'}|^2_{(x_{01}^2,z)} \right] G(x_{01}^2,z),$$

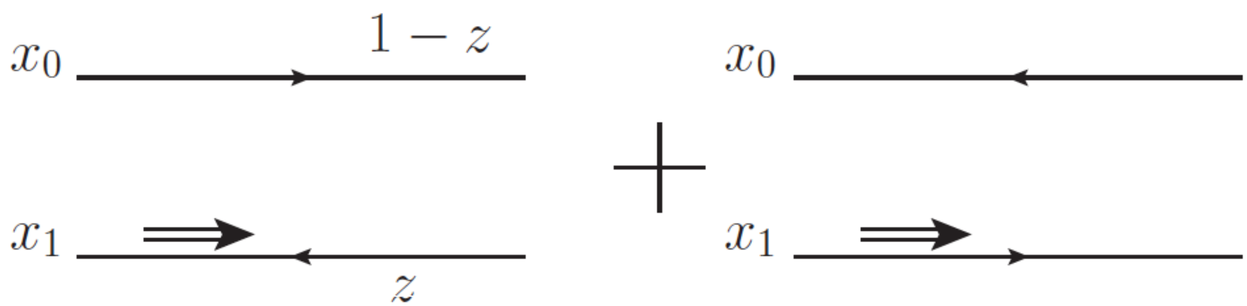
$$\Delta q^S(x,Q^2) = \frac{N_c\,N_f}{2\pi^3} \int\limits_{z_i}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2}\, G(x_{01}^2,z),$$

$$g_{1L}^S(x,k_T^2) = \frac{8\,N_c\,N_f}{(2\pi)^6} \int\limits_{z_i}^{1} \frac{dz}{z} \int d^2x_{01}\, d^2x_{0'1}\, e^{-i\underline{k}\cdot(\underline{x}_{01}-\underline{x}_{0'1})}\, \frac{\underline{x}_{01}\cdot\underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2}\, G(x_{01}^2,z)$$

- Here s is cms energy squared, $z_i=\Lambda^2/s$, $G(x_{01}^2,z) \equiv \int d^2b\, G_{10}(z)$

Polarized Dipole

- All flavor singlet small-x helicity observables depend on one object, “polarized dipole amplitude”:



$$G_{10}(z) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger}\right] + \mathrm{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger}\right] \right\rangle\!\!\right\rangle(z)$$

unpolarized quark

polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

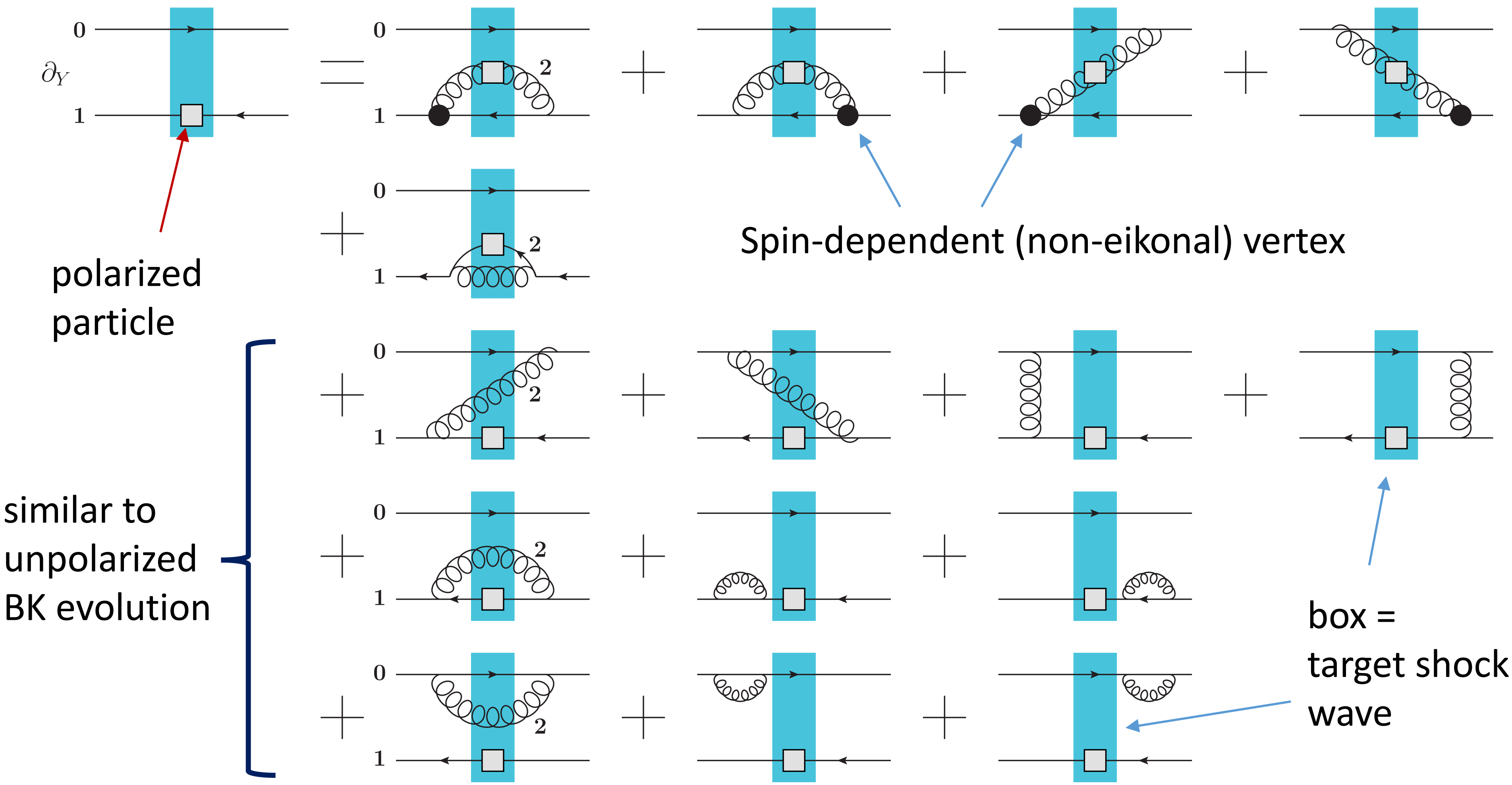
$$V_{\underline{x}} \equiv \mathcal{P} \exp\left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x})\right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle(z) \equiv zs \left\langle \mathcal{O} \right\rangle(z)$$

Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \, \ln(1/x)$$

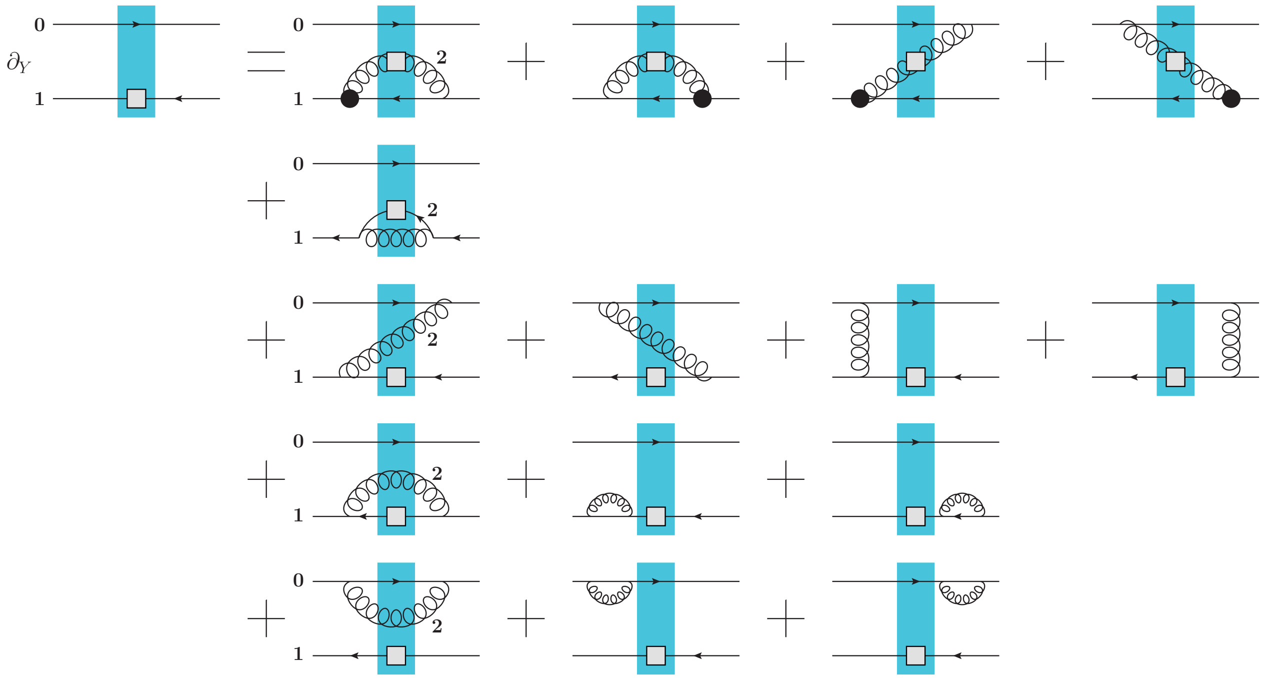
- Helicity evolution resummation parameter is double-logarithmic (DLA):

$$\alpha_s \, \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum integration being logarithmic both in UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\, s} \langle \ldots \rangle$$

$$\rho'^{\,2} = \frac{1}{z'\, s}$$

$$\frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr} \left[V_0^{unp}\, V_1^{pol\,\dagger} \right] \right\rangle\!\!\right\rangle (z) = \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr} \left[V_0^{unp}\, V_1^{pol\,\dagger} \right] \right\rangle\!\!\right\rangle_0 (z) + \frac{\alpha_s}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2}$$

$$\times \left\{ \theta(x_{10} - x_{21})\, \frac{2}{N_c} \left\langle\!\!\left\langle \mathrm{tr} \left[t^b\, V_0^{unp}\, t^a\, V_1^{unp\,\dagger} \right] \; U_2^{pol\, ba} \right\rangle\!\!\right\rangle (z') \right.$$

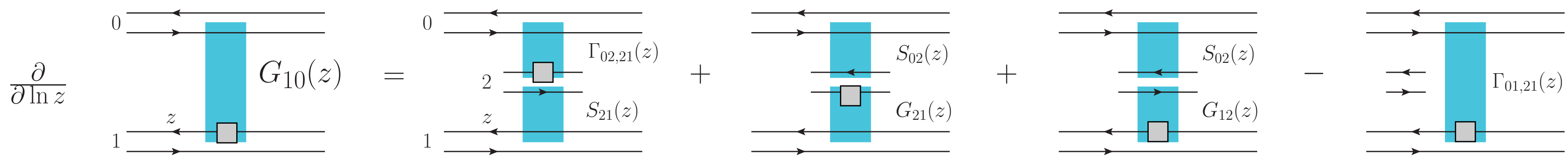
$$+ \theta(x_{10}^2 z - x_{21}^2 z')\, \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr} \left[t^b\, V_0^{unp}\, t^a\, V_2^{pol\,\dagger} \right] \; U_1^{unp\, ba} \right\rangle\!\!\right\rangle (z')$$

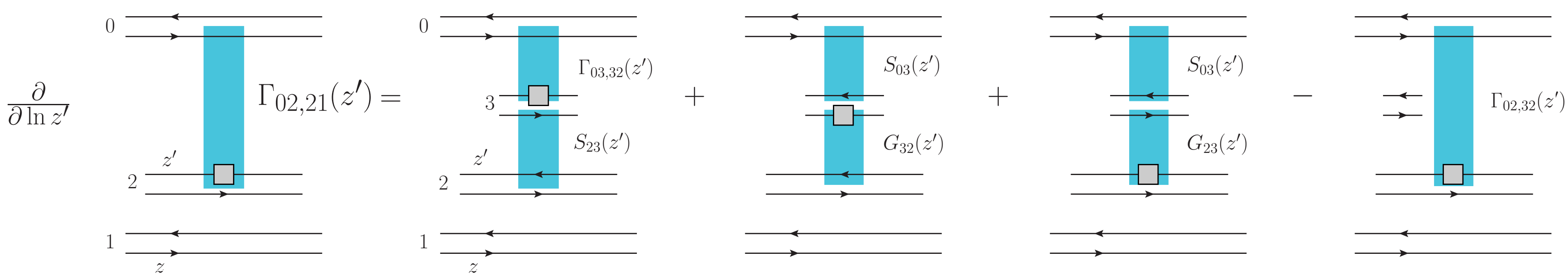
$$\left. + \theta(x_{10} - x_{21})\, \frac{1}{N_c} \left[\left\langle\!\!\left\langle \mathrm{tr} \left[V_0^{unp}\, V_2^{unp\,\dagger} \right] \; \mathrm{tr} \left[V_2^{unp}\, V_1^{pol\,\dagger} \right] \right\rangle\!\!\right\rangle (z') - N_c \left\langle\!\!\left\langle \mathrm{tr} \left[V_0^{unp}\, V_1^{pol\,\dagger} \right] \right\rangle\!\!\right\rangle (z') \right] \right\}$$

Equation does not close!

Polarized Dipole Evolution in the Large-N_c Limit

In the large-N_c limit the equations close, leading to a system of 2 equations:





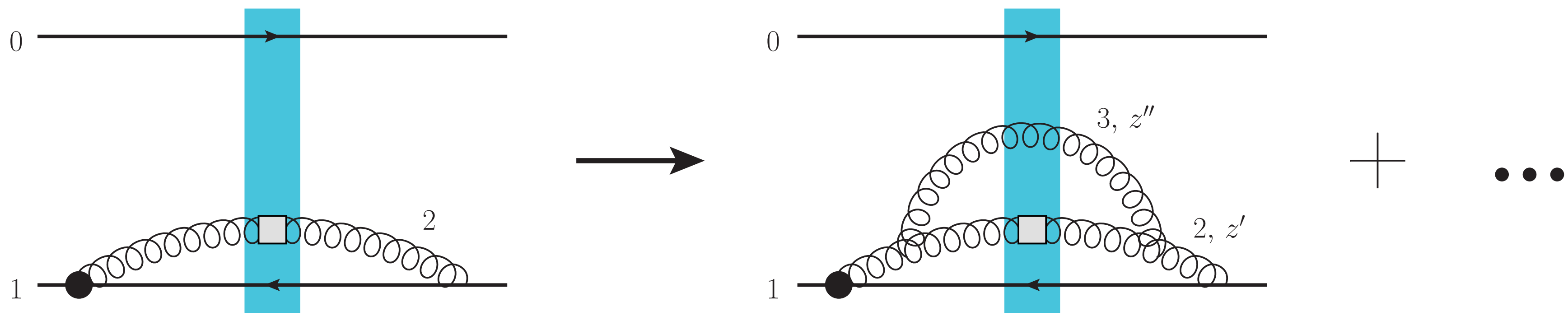
$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\,\Gamma_{02,\,21}(z')\, S_{21}(z') + 2\, G_{21}(z')\, S_{02}(z') \right.$$
$$\left. + G_{12}(z')\, S_{02}(z') - \Gamma_{01,\,21}(z') \right]$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\,\Gamma_{03,\,32}(z'')\, S_{23}(z'') + 2\, G_{32}(z'')\, S_{03}(z'') \right.$$
$$\left. + G_{23}(z'')\, S_{03}(z'') - \Gamma_{02,\,32}(z'') \right]$$

S = found from BK/JIMWLK, it is LLA

You friendly “neighborhood” dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may ’know’ about another dipole:



$$x_{21}^2\, z' \gg x_{32}^2\, z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

Large-N_c Evolution: Strict DLA Limit

- In the strict DLA limit we neglect the LLA evolution (put S=1) and get:

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z') + 3\, G(x_{21}^2, z')\right]$$

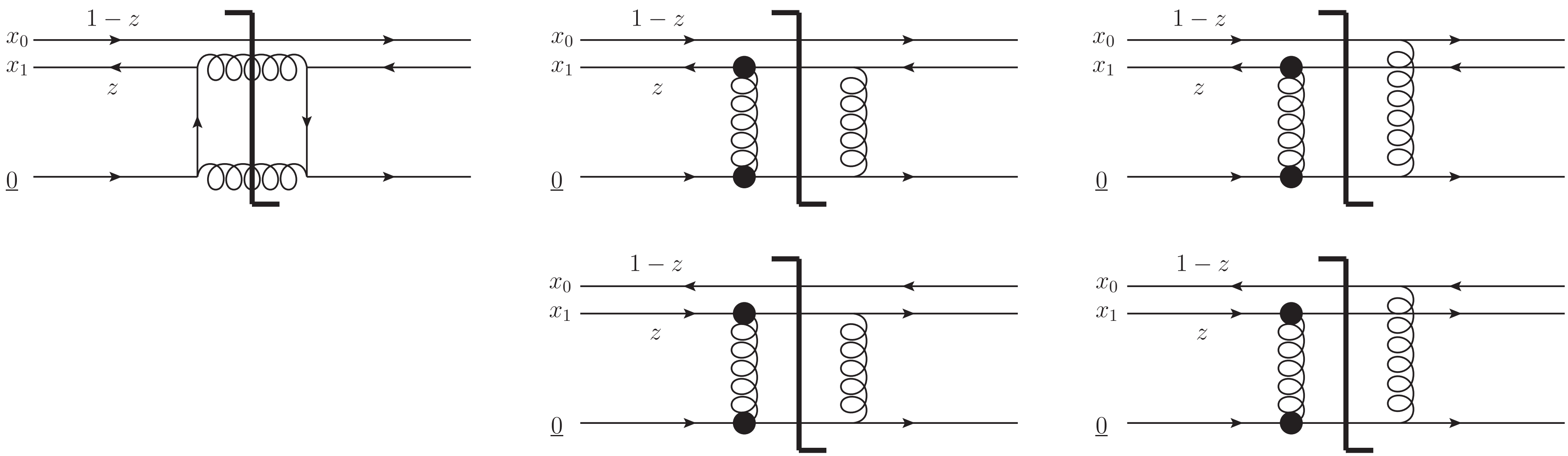
$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\right\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z'') + 3\, G(x_{32}^2, z'')\right]$$

- The initial conditions are given by Born-level graphs, for which

$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

Initial Conditions

- Initial conditions for all our evolution equations should be given by Born-level interactions (“dressed” by multiple rescatterings in the saturation case):



$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2 \ln(zs\, x_{10}^2) \right]$$

Large-N_c Equations: Numerical Solution

Yu.K., D. Pitonyak, M. Sievert, arXiv:1610.06188 [hep-ph]

Prior Results

- Small-x DLA evolution for the g_1 structure function was first considered by Bartels, Ermolaev and Ryskin (BER) in ‘96.

- Including the mixing of quark and gluon ladders, they obtained

$$\Delta\Sigma \sim g_1 \sim \left(\frac{1}{x}\right)^{z_s \sqrt{\frac{\alpha_S \, N_C}{2\,\pi}}}$$

with z_s = 3.45 for 4 quark flavors and z_S=3.66 for pure glue.

$$S_q(Q^2) = \frac{1}{2} \int\limits_0^1 dx \, \Delta\Sigma(x, Q^2)$$

- The power is large: it becomes larger than 1 for the realistic strong coupling of the order of $\alpha_s = 0.2 - 0.3$, resulting in polarized PDFs which actually grow with decreasing x fast enough for the integral of the PDFs over the low-x region to be (potentially) large (infinite).

Large-N_C Equations

- We want to find the numerical solution of the large-N_c DLA evolution equations (linearized, without saturation corrections):

$$G_{01}(z) = G_{01}^{(0)}(z) + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \, \left[\Gamma_{02,\,21}(z') + 3\, G_{21}(z') \right],$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2,\, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \, \left[\Gamma_{03,\,32}(z'') + 3\, G_{23}(z'') \right]$$

- First we define new variables:

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \, \ln \frac{zs}{\Lambda^2} \qquad\qquad s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \, \ln \frac{1}{x_{10}^2\, \Lambda^2}$$

Large-N_C Equations

- In terms of the new variables the equations become

$$G(s_{10},\eta) = G^{(0)}(s_{10},\eta) + \int\limits_{s_{10}}^{\eta} d\eta' \int\limits_{s_{10}}^{\eta'} ds_{21} \; \left[\Gamma(s_{10},s_{21},\eta') + 3\,G(s_{21},\eta')\right]$$

$$\Gamma(s_{10},s_{21},\eta') = \Gamma^{(0)}(s_{10},s_{21},\eta') + \int\limits_{s_{10}}^{\eta'} d\eta'' \int\limits_{\max\{s_{10},s_{21}+\eta''-\eta'\}}^{\eta''} ds_{32} \left[\Gamma(s_{10},s_{32},\eta'') + 3\,G(s_{32},\eta'')\right].$$

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{zs}{\Lambda^2} \qquad\qquad s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{1}{x_{10}^2\,\Lambda^2}$$

- The initial conditions are

$$G^{(0)}(s_{10},\eta) = \Gamma^{(0)}(s_{10},s_{21},\eta) = \alpha_s^2 \pi \tfrac{C_F}{N_c}\left[C_F\,\eta - 2(\eta - s_{10})\right]$$

Numerical Solution

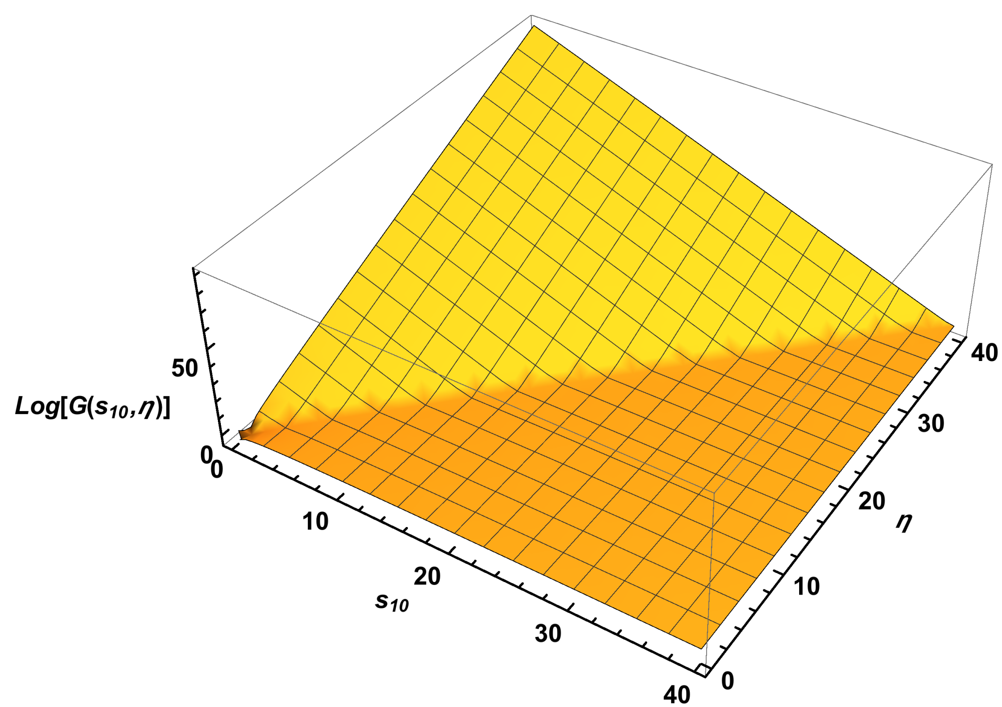
- We discretize the equations

$$G_{ij} = G_{ij}^{(0)} + \Delta\eta\, \Delta s \sum_{j'=i}^{j-1} \sum_{i'=i}^{j'} \left[\Gamma_{ii'j'} + 3\, G_{i'j'}\right],$$

$$\Gamma_{ikj} = \Gamma_{ikj}^{(0)} + \Delta\eta\, \Delta s \sum_{j'=i}^{j-1} \sum_{i'=\max\{i,k+j'-j\}}^{j'} \left[\Gamma_{ii'j'} + 3\, G_{i'j'}\right]$$

and solve them by progressively populating each fixed-η row in s.

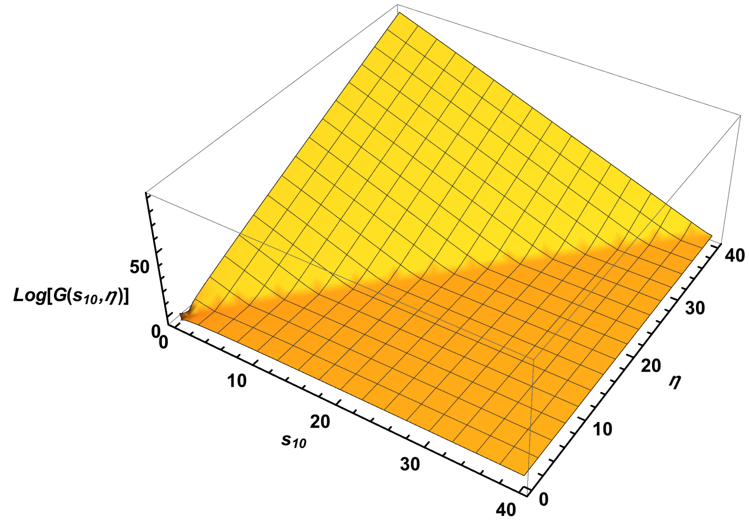
- The solution for G looks like this:



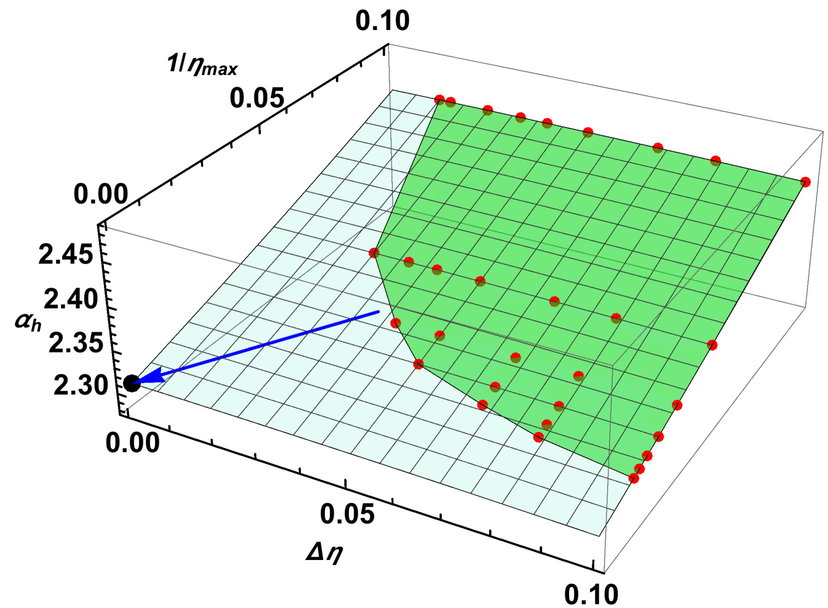
$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{zs}{\Lambda^2} \qquad s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{1}{x_{10}^2\, \Lambda^2}$$

Extracting the intercept

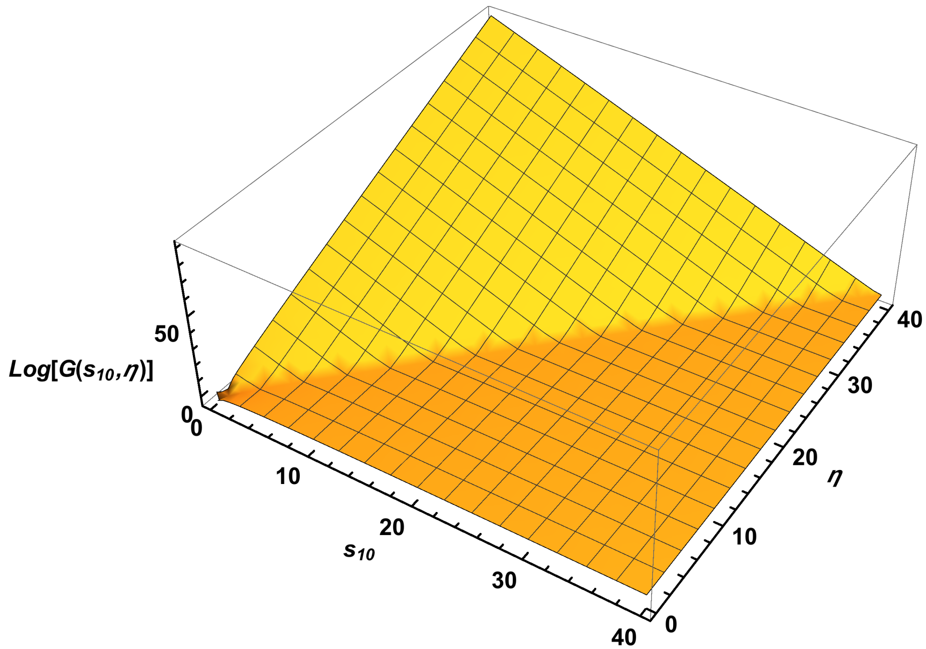
- The solution for G grows exponentially with rapidity η:



- We read off the "intercept" (the slope of ln G vs η) for different-size lattices and step sizes, and extrapolate the intercept to the continuum:



Solution of the large-N_C Equations



$$\alpha_h \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

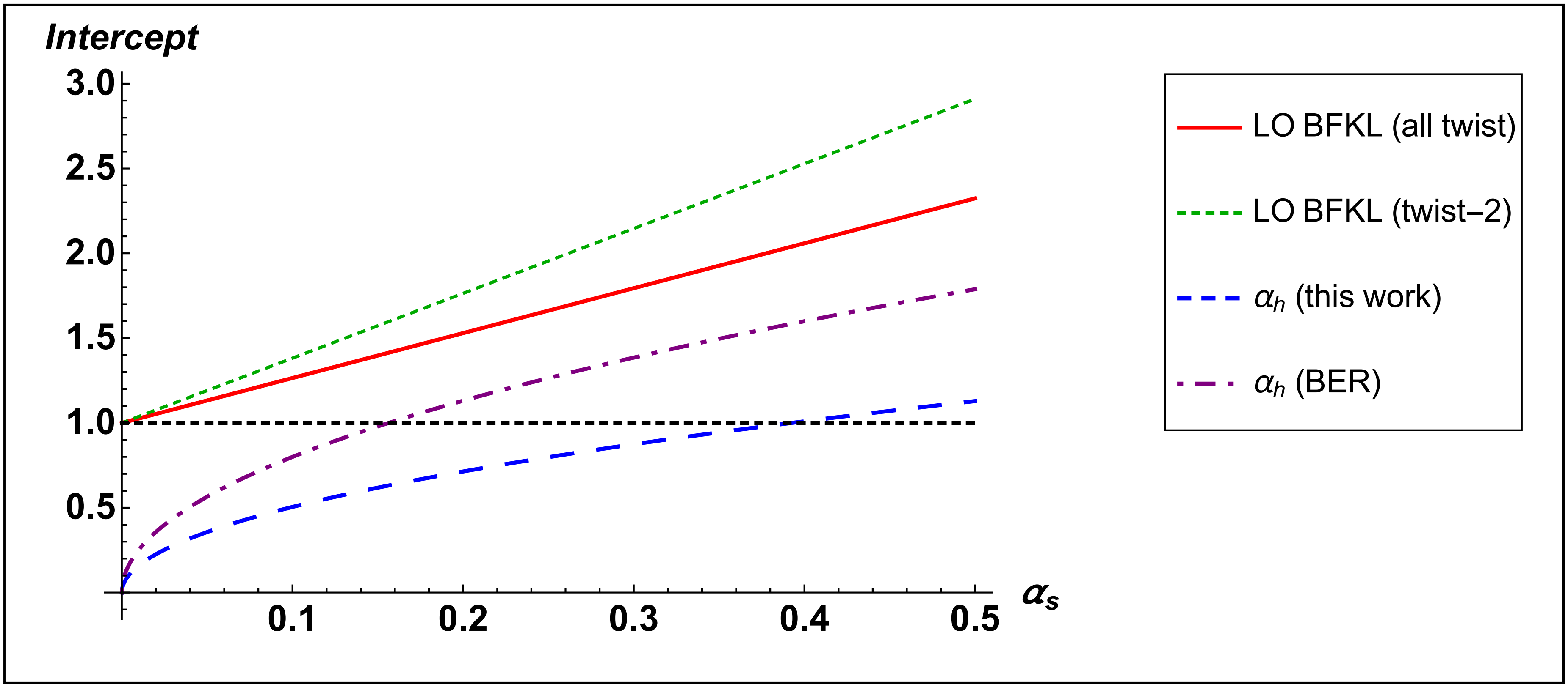
- The resulting small-x asymptotics is

$$g_1^S(x, Q^2) \sim \Delta q^S(x, Q^2) \sim g_{1L}^S(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h} \approx \left(\frac{1}{x}\right)^{2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Our result, 2.31, is about 35% smaller than BER's 3.66 any-N_C pure glue.

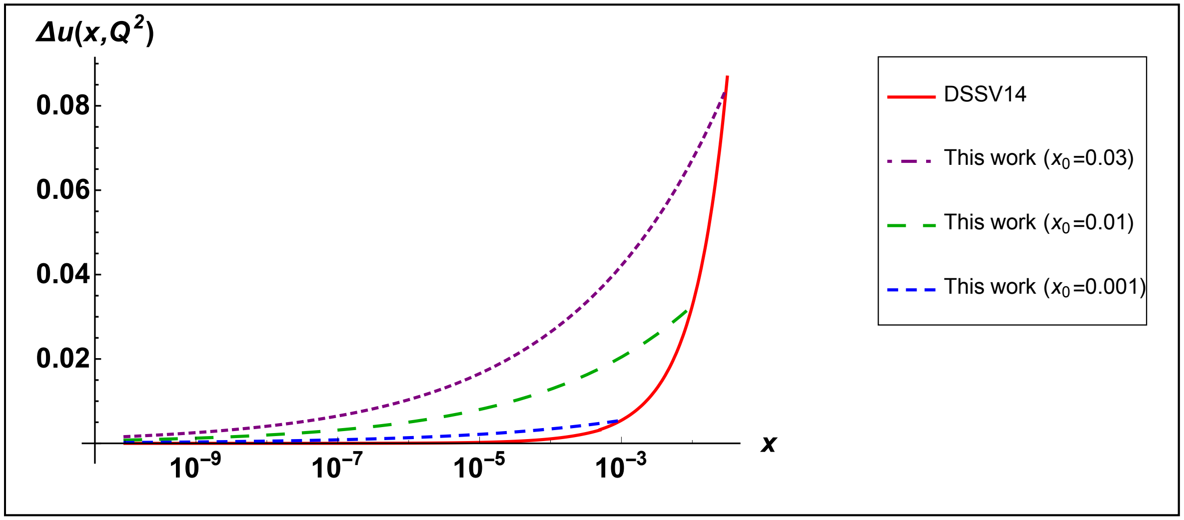
Intercepts

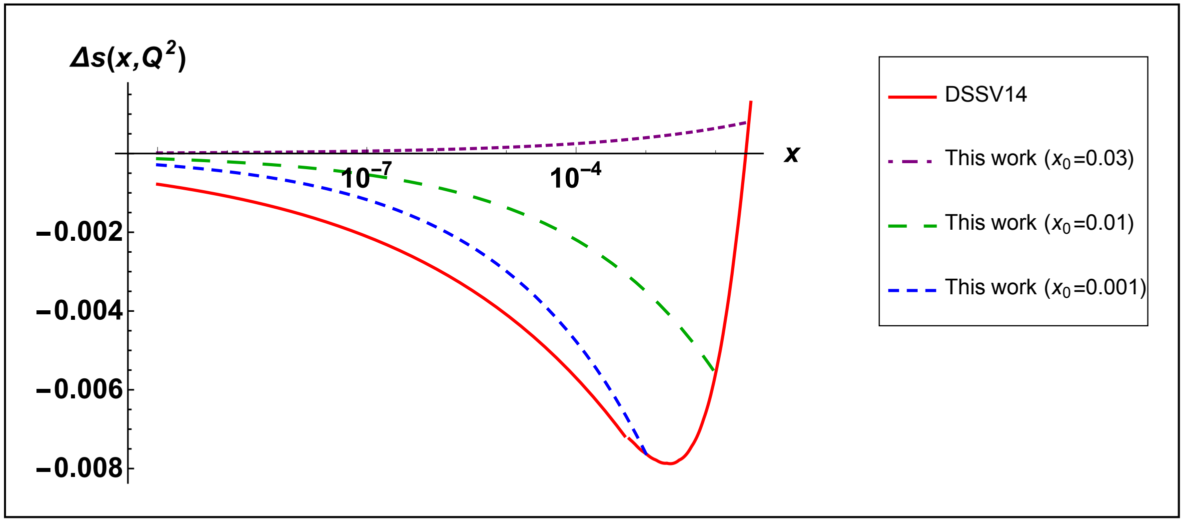
Here we plot our (flavor-singlet) helicity intercept as a function of the coupling. We show BER result and LO BFKL (all twist and leading twist) for comparison.



Impact on proton spin

- We have attached a $\Delta\tilde{\Sigma}(x,Q^2) = N\,x^{-\alpha_h}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :

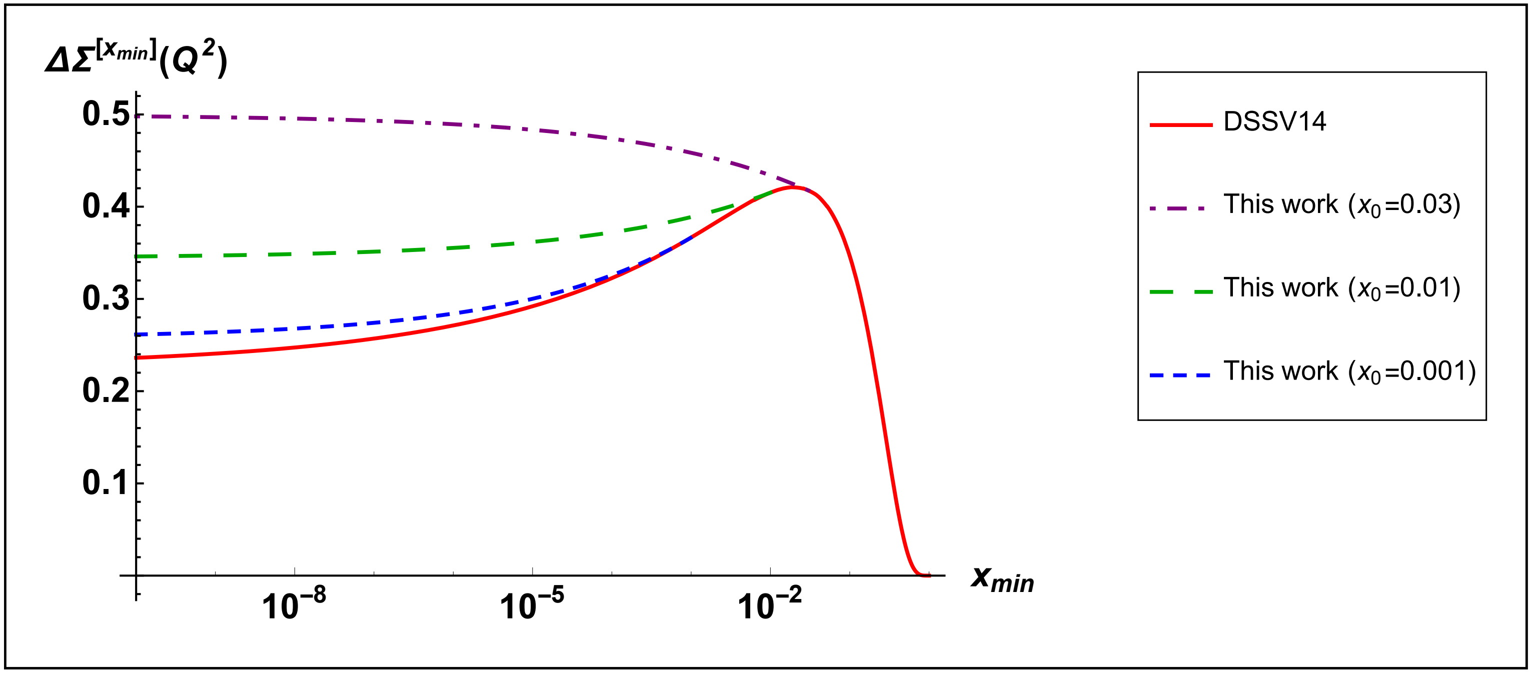




"ballpark" phenomenology

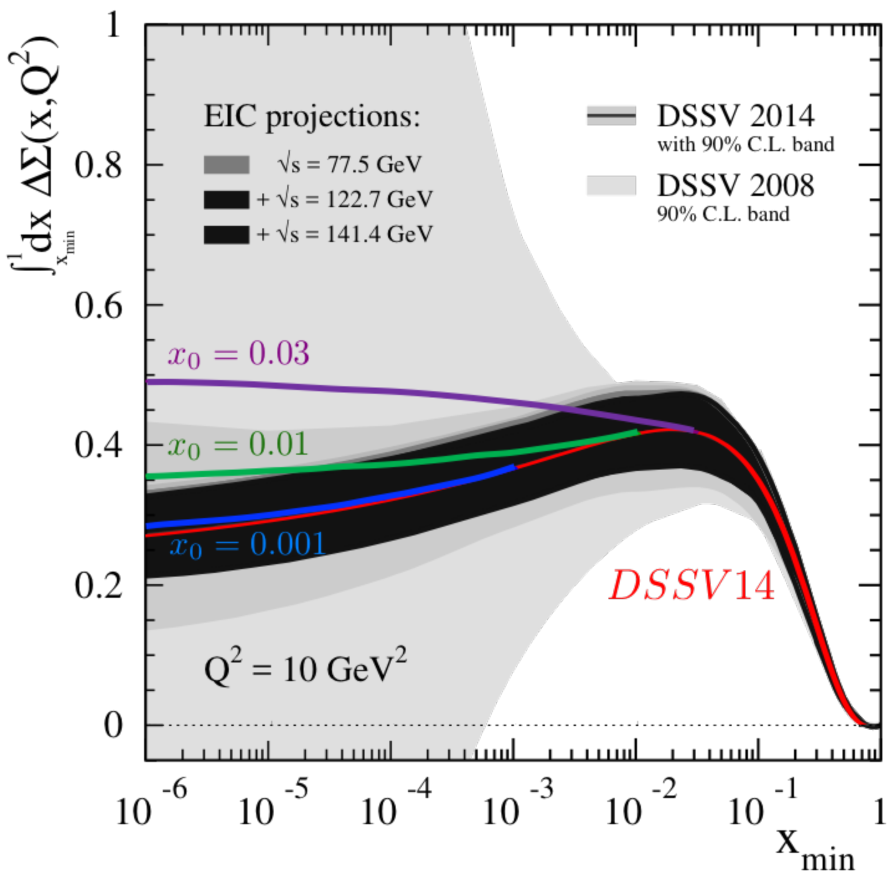
Impact on proton spin

- Defining $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta\Sigma(x, Q^2)$ we plot it for x_0=0.03, 0.01, 0.001:



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

Impact on proton spin



- Here we compare our results with DSSV, now including the error band.
- We observe consistency of our lower two curves with DSSV.
- Our upper curve disagrees with DSSV, but agrees with NNPDF (Nocera, Santopinto, '16).
- Better phenomenology is needed. EIC would definitely play a role.

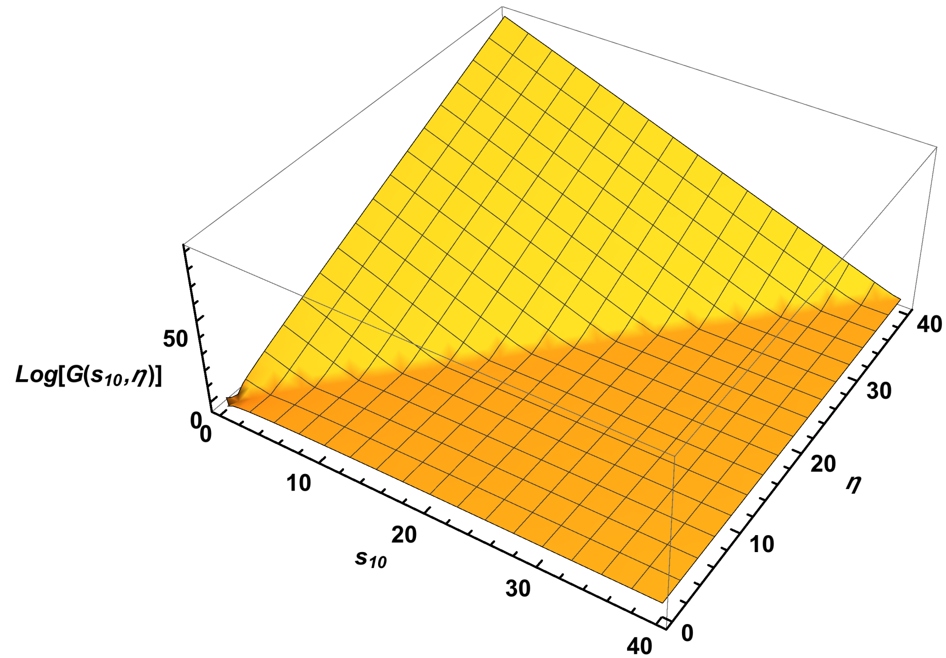
Large-N_c Equations: Analytic Solution

Yu.K., D. Pitonyak, M. Sievert, arXiv:1703.05809 [hep-ph]

Scaling

- Consider our numerical solution again

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2} \qquad s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$



- It is well approximated by

$$G(s_{10}, \eta) \propto e^{2.31\,(\eta - s_{10})}$$

- This motivated us to look for the solution in the following scaling form:

$$G(s_{10}, \eta) = G(\eta - s_{10})$$
$$\Gamma(s_{10}, s_{21}, \eta') = \Gamma(\eta' - s_{10}, \eta' - s_{21})$$

Scaling Equations

- The large-N_c evolution equations can be rewritten in terms of the scaling variables (not a trivial property, does not work for the large-N_c&N_f equations):

$$G(\zeta) = 1 + \int\limits_0^{\zeta} d\xi \int\limits_0^{\xi} d\xi' \, \left[\Gamma(\xi, \xi') + 3\, G(\xi') \right],$$

$$\Gamma(\zeta, \zeta') = 1 + \int\limits_0^{\zeta'} d\xi \int\limits_0^{\xi} d\xi' \, \left[\Gamma(\xi, \xi') + 3\, G(\xi') \right] + \int\limits_{\zeta'}^{\zeta} d\xi \int\limits_0^{\zeta'} d\xi' \left[\Gamma(\xi, \xi') + 3\, G(\xi') \right]$$

- For simplicity, pick the following initial conditions:

$$G(0) = 1, \quad \Gamma(\zeta', \zeta') = G(\zeta')$$

Analytic Solution

- These scaling equations can be solved exactly via Laplace transform + a few clever tricks, yielding

$$G(\zeta) = \int \frac{d\omega}{2\pi i}\, e^{\omega\,\zeta + \frac{\zeta}{\omega}}\, \frac{\omega^2 - 1}{\omega\,(\omega^2 - 3)},$$

$$\Gamma(\zeta, \zeta') = 4 \int \frac{d\omega}{2\pi i}\, e^{\omega\,\zeta' + \frac{\zeta}{\omega}}\, \frac{\omega^2 - 1}{\omega\,(\omega^2 - 3)} - 3 \int \frac{d\omega}{2\pi i}\, e^{\omega\,\zeta' + \frac{\zeta'}{\omega}}\, \frac{\omega^2 - 1}{\omega\,(\omega^2 - 3)}.$$

- As usual, the high-energy asymptotics is given by the right-most pole in the complex ω-plane: the pole is at $\omega = +\sqrt{3}$.

Analytic Solution and Intercept

- The contribution of the pole at $\omega = +\sqrt{3}$ is

$$G(\zeta) \approx \frac{1}{3} e^{\frac{4}{\sqrt{3}}\zeta}$$

$$\Gamma(\zeta, \zeta') \approx \frac{1}{3} e^{\frac{4}{\sqrt{3}}\zeta'} \left(4e^{\frac{\zeta-\zeta'}{\sqrt{3}}} - 3\right)$$

$$= G(\zeta') \left(4e^{\frac{\zeta-\zeta'}{\sqrt{3}}} - 3\right)$$

- The corresponding helicity intercept is

$$\alpha_h = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.3094 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- This is in complete agreement with the numerical solution!

$$\alpha_h \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Conclusions

- We have constructed new DLA evolution equations for the polarized dipole operator, which allow us to find the small-x asymptotics of the quark helicity TMDs and PDFs and of the g_1 structure function.

- Like the B-JIMWLK hierarchy, our equations do not close in general. They close in the large-N_C and large-N_C&N_f limits.

- Solution of the flavor singlet evolution equations at large-N_C gives

$$g_1^S(x,Q^2) \sim \Delta q^S(x,Q^2) \sim g_{1L}^S(x,k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h} = \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

  which may potentially generate a solid amount of spin at small-x.

- Future work includes gluon helicity TMD at small x (coming soon) and may involve including running coupling and saturation corrections + solving the large-N_C&N_f equations.

Backup Slides

Polarized "Wilson line"

- Our polarized "Wilson line" is defined as (for purely gluonic exchanges with the target)

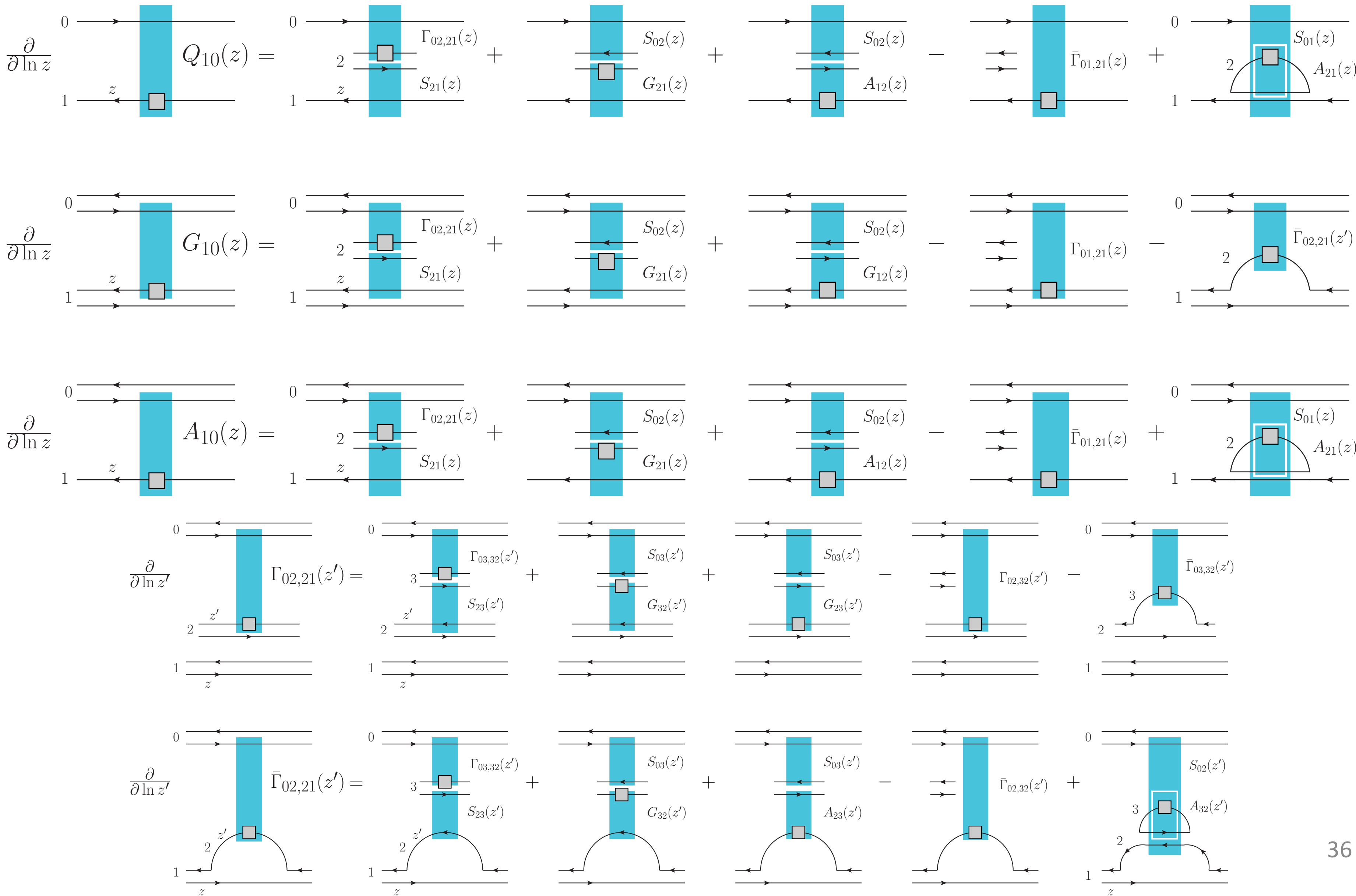
$$V_{\underline{x}}^{pol} = \frac{1}{2\,s} \int\limits_{-\infty}^{\infty} dx^- \, \mathrm{P}\exp\left\{ ig \int\limits_{x^-}^{\infty} dx'^- \, A^+(x'^-, \underline{x}) \right\} \, ig \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, \mathrm{P}\exp\left\{ ig \int\limits_{-\infty}^{x^-} dx'^- \, A^+(x'^-, \underline{x}) \right\}$$

where $\underline{A}_\Sigma(x^-, \underline{x}) = \frac{\Sigma}{2p_1^+} \, \underline{\tilde{A}}(x^-, \underline{x})$ is the spin-dependent gluon field of the plus-direction moving target. (A^+ is the unpolarized eikonal field.)

- In preparation…

Polarized Dipole Evolution in the Large-N_c&N_f Limit

In the large-N_c&N_f limit the equations close too, leading to a closed system of 5 equations:



Large-N_c&N_f Evolution

- The evolution equations read (in the strict DLA limit, S=1):

$$Q_{01}(z) = Q_{01}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10} - x_{21}) \, \left[G_{12}(z') + \Gamma_{02,\,21}(z') + A_{21}(z') - \bar{\Gamma}_{01,\,21}(z') \right]$$

$$+ \frac{\alpha_s N_c}{4\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10}^2 z - x_{21}^2 z') \, A_{21}(z'),$$

$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10} - x_{21}) \, \left[\Gamma_{02,\,21}(z') + 3\, G_{12}(z') \right]$$

$$- \frac{\alpha_s N_f}{4\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10}^2 z - x_{21}^2 z') \, \bar{\Gamma}_{02,\,21}(z'),$$

$$A_{01}(z) = A_{01}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10} - x_{21}) \, \left[G_{12}(z') + \Gamma_{02,\,21}(z') + A_{21}(z') - \bar{\Gamma}_{01,\,21}(z') \right]$$

$$+ \frac{\alpha_s N_c}{4\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10}^2 z - x_{21}^2 z') \, A_{12}(z').$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \, \left[\Gamma_{03,\,32}(z'') + 3\, G_{23}(z'') \right]$$

$$- \frac{\alpha_s N_f}{4\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{x_{21}^2\, z'/z''} \frac{dx_{32}^2}{x_{32}^2} \, \bar{\Gamma}_{03,\,32}(z'),$$

$$\bar{\Gamma}_{02,\,21}(z') = \bar{\Gamma}_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \, \left[\Gamma_{03,\,32}(z'') + G_{23}(z'') + A_{23}(z'') - \bar{\Gamma}_{02,\,32}(z'') \right]$$

$$+ \frac{\alpha_s N_c}{4\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{x_{21}^2\, z'/z''} \frac{dx_{32}^2}{x_{32}^2} \, A_{32}(z').$$

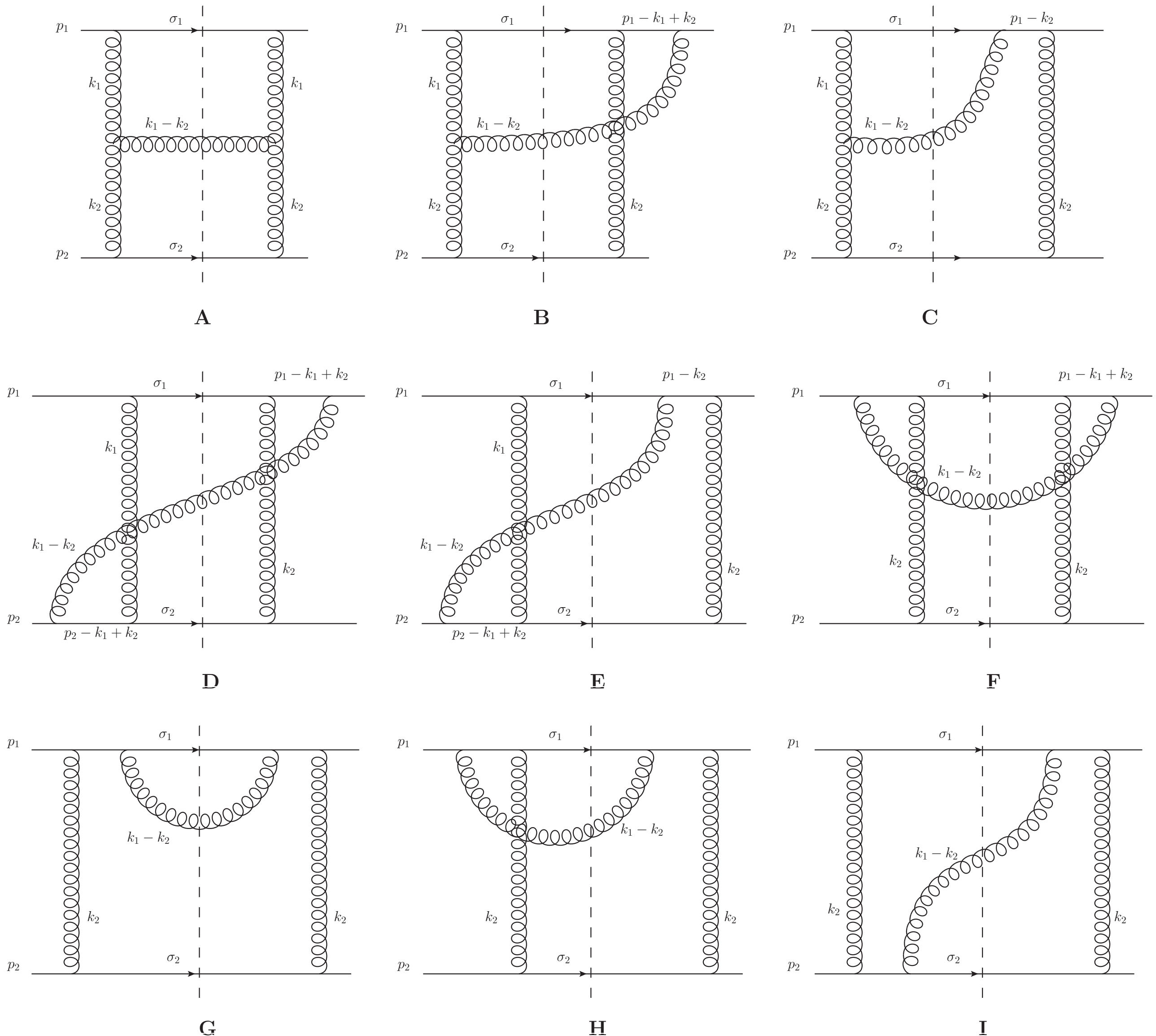
Intercepts

- We can summarize some LO intercepts, including the ones we found, in the following table:

Observable	Evolution	Intercept	$Q^2 = 3\ \mathrm{GeV}^2$ $\alpha_s = 0.343$	$Q^2 = 10\ \mathrm{GeV}^2$ $\alpha_s = 0.249$	$Q^2 = 87\ \mathrm{GeV}^2$ $\alpha_s = 0.18$
Unpolarized flavor singlet structure function F_2	LO BFKL Pomeron	$1 + \frac{\alpha_s N_c}{\pi} 4 \ln 2$	1.908	1.659	1.477
Unpolarized flavor non-singlet structure function F_2	Reggeon	$\sqrt{\frac{2\,\alpha_s C_F}{\pi}}$	0.540	0.460	0.391
Flavor singlet structure function g_1^S	us (Pure Glue)	$2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}$	0.936	0.797	0.678
	BER (Pure Glue)	$3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.481	1.262	1.073
	BER ($N_f = 4$)	$3.45\sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.400	1.190	1.011
Flavor non-singlet structure function g_1^{NS}	BER and us (large-N_c)	$\sqrt{\frac{\alpha_s N_c}{\pi}}$	0.572	0.488	0.415

$$\alpha_h^{BER} = \sqrt{\frac{17+\sqrt{97}}{2}}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Comparison with BER



To better understand BER work, we tried calculating one (real) step of DLA helicity evolution for the qq->qq scattering.

It appears that we have identified the k_2>> k_1 (or k_1>> k_2) regime in which diagrams A, B, C, D, E, I are DLA, which was not considered by BER for B, C, … I.

Helicity Evolution at Small x

flavor non-singlet case

Yu.K., D. Pitonyak, M. Sievert, arXiv:1610.06197 [hep-ph]

Flavor Non-Singlet Observables

- In the flavor non-singlet case, all helicity observables again depend on the polarized dipole amplitude:

$$g_1^{NS}(x, Q^2) = \frac{N_c}{2\,\pi^2 \alpha_{EM}} \int\limits_{z_i}^{1} \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2} \sum_{\lambda\sigma\sigma'} |\psi^T_{\lambda\sigma\sigma'}|^2_{(x_{01}^2, z)} + \sum_{\sigma\sigma'} |\psi^L_{\sigma\sigma'}|^2_{(x_{01}^2, z)} \right] G^{NS}(x_{01}^2, z),$$

$$\Delta q^{NS}(x, Q^2) = \frac{N_c}{2\pi^3} \int\limits_{z_i}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2} G^{NS}(x_{01}^2, z),$$

$$g_{1L}^{NS}(x, k_T^2) = \frac{8\,N_c}{(2\pi)^6} \int\limits_{z_i}^{1} \frac{dz}{z} \int d^2 x_{01}\, d^2 x_{0'1}\, e^{-i\underline{k}\cdot(\underline{x}_{01} - \underline{x}_{0'1})} \, \frac{\underline{x}_{01} \cdot \underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2} \, G^{NS}(x_{01}^2, z)$$

- Polarized dipole amplitude is different (difference instead of sum):

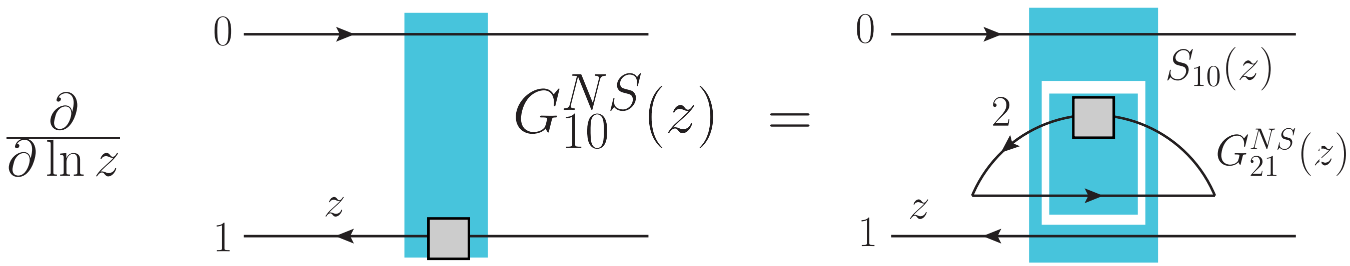
$$G_{10}^{NS}(z) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger} \right] - \mathrm{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle (z)$$

- This is related to the definition

$$\Delta q^{NS}(x, Q^2) \equiv \Delta q^f(x, Q^2) - \Delta \bar{q}^f(x, Q^2)$$

Flavor Non-Singlet Evolution

- Evolution equations end up being much simpler in the non-singlet case:



$$G_{10}^{NS}(z) = G_{10}^{NS\,(0)}(z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2 \frac{z}{z'}} \frac{dx_{21}^2}{x_{21}^2} \, S_{10}(z') \, G_{21}^{NS}(z')$$

- Analytical solution (in the DLA case, S=1) leads to (in agreement with Bartels et al, '95)

$$g_1^{NS}(x, Q^2) \sim \Delta q^{NS}(x, Q^2) \sim g_{1L}^{NS}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^{NS}} \approx \left(\frac{1}{x}\right)^{\sqrt{\frac{\alpha_s N_c}{\pi}}}$$

- The resulting intercept is smaller than the flavor-singlet intercept.