

CONSIDERATION OF THE USE OF FEEDBACK IN A
TRAVELING-WAVE SUPERCONDUCTING ACCELERATOR*1

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I. INTRODUCTION

In a standing-wave accelerator structure, essentially all of the input rf power is inherently utilized (assuming proper input matching) to set up the accelerating fields and for conversion to beam power. Because of this basic simplicity all of the experimental work on superconducting accelerators carried out to date has employed the standing-wave structure. The theoretical performance of the standing-wave superconducting accelerator under beam loading conditions has been studied by Wilson and Schwettman.² (See Section VI of this report for further discussion.) The energy gain in a properly matched standing-wave (SW) accelerator with negligible beam loading is given by³

$$V_{SW} = \left(\frac{\tanh \tau/2}{\tau/2} \right)^{\frac{1}{2}} (1 + e^{-2\tau})^{-\frac{1}{2}} (P_s r_o \ell)^{\frac{1}{2}}, \quad (1)$$

where P_s is the rf power from the source, r_o is the shunt impedance per unit length, ℓ is the length of the accelerator structure, and τ is the attenuation parameter in nepers. For superconducting accelerators, where τ is very small, Eq. (1) becomes:

$$V_{SW} \approx \left(\frac{P_s r_o \ell}{2} \right)^{\frac{1}{2}} = \left(\frac{P_d r_o \ell}{2} \right)^{\frac{1}{2}}, \quad (2)$$

where $P_d = P_s$ is the power dissipated in the accelerator structure.

The energy gain in the traveling-wave (TW) accelerator with negligible beam loading is given by

$$V_{TW} = (2\tau)^{\frac{1}{2}} \left(\frac{1 - e^{-\tau}}{\tau} \right) (P_s r_o \ell)^{\frac{1}{2}}. \quad (3)$$

When τ is very small, Eq. (3) becomes:

$$V_{TW} \approx (2\tau P_s r_o \ell)^{\frac{1}{2}} = (P_d r_o \ell)^{\frac{1}{2}}, \quad (4)$$

*Work supported by the U.S. Atomic Energy Commission.

1. This report is based upon an earlier Stanford Linear Accelerator Center Technical Note: R.B. Neal, SLAC-TN-68-1 (1968).
2. P.B. Wilson and H.A. Schwettman, IEEE Trans. Nucl. Sci. NS-12, No. 3, 1045 (1965).
3. G.A. Loew and R.B. Neal, in Linear Accelerators, edited by P. Lapostolle and A. Septier (North-Holland Publishing Company, Amsterdam, The Netherlands) (to be published).

where $2\pi P_s = P_d$ is the power dissipated in the accelerator structure. In this case, P_d , which is the power useful in setting up the accelerating fields, is very low and most of the rf power is lost at the output end of the accelerator. To make full use of the available power it is necessary to feed back the residual rf power through an external loop and to combine it in proper phase with the input power. If this is done, the power and fields in the TW structure will build up to a very high level. If the loss in the external loop is negligible compared to the loss in the accelerator structure, essentially all of the input power is available for setting up the accelerating fields, i.e., $P_d \approx P_s$. Comparing Eqs. (2) and (4), it is noted that, for the same no-load energy gain, the power dissipated in the SW structure must be twice the power dissipated in the TW structure with feedback, assuming the structures have the same lengths and shunt impedances. In actual fact, it is possible to compensate largely for this disadvantage in the SW case by using the π mode or the $\pi/2$ mode in a bi-periodic structure for which the shunt impedances are considerably increased.

From another viewpoint, for a given net energy gain, the ratio of peak to average fields in the standing-wave structure is up to two times as high as this ratio in the traveling-wave structure. This consideration gives an advantage to the traveling-wave structure for superconducting accelerator applications. Again, the relative advantage of the TW structure is reduced when the π mode or the special $\pi/2$ bi-periodic mode is used in the SW structure.

The original idea of using feedback in conjunction with a traveling-wave linear accelerator was proposed by R.-Shersby-Harvie and Mullett⁴ in 1949. This method was used on a number of early low energy British accelerators designed for medical therapy. The use of the feedback principle in resonant rings for the purpose of testing various microwave components such as rf windows was demonstrated by Milosevic and Vautey.⁵ Hahn and Halama^{6,7} have studied the possibility of using the resonant ring concept in superconducting rf beam separators.

II. OPTIMUM FEEDBACK IN THE TW ACCELERATOR WITH BEAM LOADING

A schematic of the TW accelerator with feedback is shown in Fig. 1. The residual rf power at the end of the accelerator is fed back to the input end where it is combined with the source power P_s by means of a suitable waveguide bridge. The combined power P_o is then fed into the accelerator. The bridge ratio, which will be designated by the symbol g , is defined as the ratio of the powers which the bridge is designed to combine. When the ratio of the feedback power to the source power is equal to the bridge ratio g and when the feedback phase is properly adjusted, the power input to the accelerator will be $(1 + g)$ times the source power and the power P_L to the external load will be zero.

Suppose that the attenuation due to beam loading and to wall losses in the accelerator structure and feedback loop is such that $P_F = P_o/x^2$. Then the condition for maximum power input to the accelerator and zero power to the resistive load is

$$\frac{P_s(1+g)}{x^2} = g P_s$$

4. R.B. R.-Shersby-Harvie and L.B. Mullett, Proc. Phys. Soc. London B62, 270 (1949).
5. L.J. Milosevic and R. Vautey, IRE Trans. MTT-6, 136 (1958).
6. H.J. Halama and H. Hahn, IEEE Trans. Nucl. Sci. NS-14, No. 3, 350 (1967).
7. H. Hahn and H.J. Halama, IEEE Trans. Nucl. Sci. NS-14, No. 3, 356 (1967).

or

$$g = \frac{1}{x^2 - 1} \quad (5)$$

When the condition of Eq. (5) is met, the steady-state power buildup ratio in the accelerator will be:

$$\frac{P_o}{P_s} = 1 + g = \frac{1}{1 - (1/x^2)} \quad (6)$$

From Eq. (6), it is noted that a large buildup ratio results when x^2 is small (close to 1), i.e., when a large fraction of the input rf power is fed back to the bridge. The design value of the bridge ratio g must be correspondingly high as also given by Eq. (6).

In the presence of beam loading, the residual power at the output ($z = \ell$) of the accelerator section is⁸:

$$\frac{P_\ell}{P_o} = \left[e^{-\tau} - \left(\frac{i^2 r_o \ell}{P_o} \right)^{\frac{1}{2}} \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} \right]^2 \quad (7)$$

where i is the peak beam current and the other terms are as previously defined. The power P_F fed back to the bridge is $P_\ell e^{-2\gamma}$, where γ is the attenuation in the feedback loop expressed in nepers. Thus,

$$\frac{1}{x^2} = \frac{P_F}{P_o} = \left[e^{-(\tau+\gamma)} - e^{-\gamma} \left(\frac{i^2 r_o \ell}{P_o} \right)^{\frac{1}{2}} \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} \right]^2 \quad (8)$$

Inserting $1/x^2$ from Eq. (8) in Eq. (6) and solving for P_o/P_s yields the result:

$$\left(\frac{P_o}{P_s} \right)^{\frac{1}{2}} = \frac{1 + i_n^2 e^{-2\gamma} \frac{(1 - e^{-\tau})^2}{2\tau}}{i_n e^{-(\tau+2\gamma)} \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} + \left[(1 - e^{-2(\tau+\gamma)}) + i_n^2 e^{-2\gamma} \frac{(1 - e^{-2\tau})^2}{2\tau} \right]^{\frac{1}{2}}} \quad (9)$$

where

$$i_n = \left(\frac{i^2 r_o \ell}{P_s} \right)^{\frac{1}{2}}$$

The normalized beam energy with beam loading is given by⁸:

$$V_n = (2\tau)^{\frac{1}{2}} \left(\frac{1 - e^{-\tau}}{\tau} \right) \left(\frac{P_o}{P_s} \right)^{\frac{1}{2}} - i_n \left(1 - \frac{1 - e^{-\tau}}{\tau} \right) \quad (10)$$

where

$$V_n = \frac{V}{(P_s \ell r_o)^{\frac{1}{2}}}$$

8. R.B. Neal, J. Appl. Phys. 29, 1019 (1958).

Substituting Eq. (9) in Eq. (10) yields the result:

$$V_n = \frac{1 + i_n^2 e^{-2\gamma} \frac{(1 - e^{-\tau})^2}{2\tau}}{\frac{i_n e^{-(\tau+2\gamma)}}{2} + \left[\frac{\tau}{2} \frac{1 - e^{-2(\tau+\gamma)}}{(1 - e^{-\tau})^2} + \frac{i_n^2 e^{-2\gamma}}{4} \right]^{\frac{1}{2}}} - i_n \left(1 - \frac{1 - e^{-\tau}}{\tau} \right) \quad (11)$$

The beam conversion efficiency η is defined as the fraction of the power from the rf source which is converted into beam power at steady state, i.e.,

$$\eta = \frac{V_i}{P_s} \quad (12)$$

From the definitions of V_n and i_n , it is clear that η can also be expressed as $\eta = V_n i_n$. Thus, η may be obtained by multiplying both sides of Eq. (11) by i_n . Two sets of curves of V_n and η versus i_n are shown in Fig. 2. One set is based on $\gamma/\tau = 10^4$ [case (a)] which is typical of feedback through a loop which is at room temperature while the accelerator structure is supercooled. The other set is based on $\gamma/\tau = 0.1$ [case (b)] which is a rough approximation* for the case where the feedback loop as well as the accelerator structure is supercooled. From a comparison of these curves, several observations may be made. For fixed rf power input, the theoretical no-load energy is about 95 times higher for case (b) where $\gamma/\tau = 0.1$ than for case (a) where $\gamma/\tau = 10^4$. As the beam current increases, η increases much more rapidly for case (b) than for case (a) and approaches 100% for relatively small values of current. Similarly, the beam energy for case (b) drops off much more rapidly with increasing current. At higher values of beam current, the values of beam energy and η for both cases approach the same values. Case (b) is obviously superior where the attainment of high energy is paramount. However, the very large value of the beam loading derivative requires that the current be maintained constant with high accuracy in order to achieve energy stability.

III. FIXED BRIDGE RATIO g IN THE TW ACCELERATOR WITH BEAM LOADING

The above discussion is based upon optimum feedback which implies a bridge ratio g which can be varied to suit any degree of beam loading. While variable ratio bridges have been designed,⁹ they are expensive and cumbersome and are of questionable feasibility at cryogenic temperatures. A bridge having a fixed ratio corresponding to a specific value of design current may therefore be required. However, it is desirable to understand the performance of such a system over the entire range of feasible beam currents. To gain insight, it is helpful to study the transient buildup of a feedback system with fixed bridge ratio g .

The power buildup process is a stepwise affair with intervals between steps equal to the loop transit time. Initially, the voltage V_s applied to the bridge from the power source will divide, producing a voltage $V_s/(1 + g)^{\frac{1}{2}}$ in the accelerator arm and a

* At room temperatures, the attenuation in nepers per unit length is typically 50 to 100 times as high in the accelerator structure as in a similar length of ordinary waveguide. Because of the presence of bends and the recombining bridge in the feedback loop, the more conservative estimate of 10 is being used in this example.

9. C.F. Bareford and M.G. Kelliher, Philips Tech. Rev. 15, No. 1, 11. 1-26 (1953).

voltage $V_s [g/(1+g)]^{\frac{1}{2}}$ in the resistive load arm. After one transit through the accelerator feedback loop, a voltage $V_s/[x(1+g)^{\frac{1}{2}}]$ appears in the feedback arm. If the phase is correct, this feedback voltage divides, the fraction $[g/(1+g)]^{\frac{1}{2}}$ being added to the existing voltage in the accelerator arm and the fraction $[1/(1+g)]^{\frac{1}{2}}$ being subtracted from the existing voltage in the load arm. Additional transits lead to the series

$$\begin{aligned} \frac{V_o}{V_s} &= \frac{1}{(1+g)^{\frac{1}{2}}} \left[1 + \left(\frac{1}{x}\right) \left(\frac{g}{1+g}\right)^{\frac{1}{2}} + \left(\frac{1}{x}\right)^2 \frac{g}{1+g} + \dots \right] \\ &= \frac{x}{x(1+g)^{\frac{1}{2}} - g^{\frac{1}{2}}} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \frac{V_L}{V_s} &= \left(\frac{g}{1+g}\right)^{\frac{1}{2}} - \left[\frac{1}{x(1+g)}\right] \left[1 + \left(\frac{1}{x}\right) \left(\frac{g}{1+g}\right)^{\frac{1}{2}} + \left(\frac{1}{x}\right)^2 \frac{g}{1+g} + \dots \right] \\ &= \frac{xg^{\frac{1}{2}} - (1+g)^{\frac{1}{2}}}{x(1+g)^{\frac{1}{2}} - g^{\frac{1}{2}}} \end{aligned} \quad (14)$$

The squares of Eqs. (13) and (14) give the steady-state values of P_o/P_s and P_L/P_s , respectively. From Eq. (14) it is noted that when $xg^{\frac{1}{2}} = (1+g)^{\frac{1}{2}}$, i.e., $g = 1/(x^2 - 1)$, V_L and $P_L = 0$, and all of the power is delivered to the accelerator. In this case $P_o/P_s = 1 + g$ as given in Eq. (6) for the optimum feedback case.

Substituting x from Eq. (8) in Eqs. (13) and (14), squaring and solving for P_o/P_s and P_L/P_s yields the results:

$$\left(\frac{P_o}{P_s}\right)^{\frac{1}{2}} = \frac{1 - g^{\frac{1}{2}} e^{-\gamma} i_n \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}}}{(1+g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)}} \quad (15)$$

$$\left(\frac{P_L}{P_s}\right)^{\frac{1}{2}} = \frac{g^{\frac{1}{2}} - (1+g)^{\frac{1}{2}} e^{-(\tau+\gamma)} + e^{-\gamma} i_n \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}}}{(1+g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)}} \quad (16)$$

Substituting $(P_o/P_s)^{\frac{1}{2}}$ from Eq. (15) in the basic energy equation [Eq. (10)] gives the normalized beam energy:

$$(V_n)_{g \text{ fixed}} = (2\tau)^{\frac{1}{2}} \frac{1 - e^{-\tau}}{\tau} \left[\frac{1 - g^{\frac{1}{2}} e^{-\gamma} i_n \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}}}{(1+g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)}} \right] - i_n \left(1 - \frac{1 - e^{-\tau}}{\tau} \right) \quad (17)$$

The conversion efficiency η is given, as previously stated, by the product of V_n from Eq. (17) by i_n . Unlike the case with optimum feedback where η continues to approach unity as i_n increases, there is, for fixed bridge ratio g , a value of beam current which results in maximum η . Moreover, for each value of i_n , there is a value of the bridge ratio g which results in the maximum values of V_n and η at that i_n . Maximizing V_n in Eq. (17) with respect to g (holding i_n constant) gives the optimizing relationship:

$$(1 + g)^{\frac{1}{2}} e^{-(\tau+\gamma)} - g^{\frac{1}{2}} = e^{-\gamma} \left[\frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} \right] i_n . \quad (18)$$

Equation (18) may be used to calculate the optimum design value of g for given values of the parameters τ , γ , and i_n .

When η [obtained by multiplying V_n from Eq. (17) by i_n] is maximized with respect to i_n , the necessary condition is found to be:

$$(i_n) \eta_{\max} = \frac{\left(\frac{\tau}{2} \right)^{\frac{1}{2}} \left(\frac{1 - e^{-\tau}}{\tau} \right)}{\frac{(1 - e^{-\tau})^2}{\tau} g^{\frac{1}{2}} e^{-\gamma} + \left[(1 + g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)} \right] \left[1 - \left(\frac{1 - e^{-\tau}}{\tau} \right) \right]} . \quad (19)$$

When τ and γ are small, Eq. (19) reduces to the simple expression:

$$(i_n) \eta_{\max} \approx \frac{1}{(2g\tau)^{\frac{1}{2}}} . \quad (20)$$

Substituting the condition given by Eq. (19) in the equation for η yields the expression for maximum conversion efficiency:

$$\eta_{\max} = \frac{1}{\left[(1 + g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)} \right] \left\{ 2g^{\frac{1}{2}} e^{-\gamma} + 2 \left[(1 + g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)} \right] \left[\frac{\tau - (1 - e^{-\tau})}{(1 - e^{-\tau})^2} \right] \right\}} . \quad (21)$$

When τ and γ are both small, Eq. (21) reduces to the expression:

$$\eta_{\max} \approx \frac{1}{1 + 2g(\tau + \gamma)} . \quad (22)$$

When $(i_n) \eta_{\max}$ from Eq. (19) is substituted in the expression for beam energy [Eq. (17)] the result is

$$(V_n) \eta_{\max} = \frac{\left(\frac{\tau}{2} \right)^{\frac{1}{2}} \frac{1 - e^{-\tau}}{\tau}}{(1 + g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)}} = \frac{V_{no}}{2} , \quad (23)$$

where V_{no} is the no-load energy which can be obtained by setting $i_n = 0$ in Eq. (17). Thus, at maximum conversion efficiency, the beam energy is reduced to one-half of the no-load energy as in the case of the simple single feed accelerator without feedback.⁸ From the form of Eq. (17), it is clear that V_n decreases linearly as the beam current increases. In general, as the beam current is varied the beam energy can be expressed as follows:

$$V_n = V_{no} \left(1 - \frac{i_n}{2(i_n) \eta_{\max}} \right) . \quad (24)$$

When τ and γ are small, Eq. (23) becomes:

$$(V_n) \eta_{\max} \approx \frac{\left(\frac{\tau}{2}\right)^{\frac{1}{2}}}{g^{\frac{1}{2}} \left[\frac{1}{2g} + (\tau + \gamma) \right]} \quad (25)$$

The normalized beam energy V_n and the beam conversion efficiency η are shown versus i_n in Fig. 3 for three values of the bridge ratio g ($g = 10^4$, 10^5 , and 9.08×10^5). The latter value of g calculated from Eq. (18) gives a maximum no-load energy for the assumed values of τ and γ . The same values of τ and γ have been taken as for case (b) of Fig. 2. It is noted that a large bridge ratio results in a high value of the no-load energy and a high value of the beam loading derivative. Also, for large g , the conversion efficiency peaks at a lower value of beam current and the maximum efficiency is less than for smaller values of g . The dashed curves in Fig. 3 show the values of V_n and η for the optimum feedback case as given by Eq. (11) for V_n and by Eq. (11) multiplied by i_n for η . These dashed curves are the envelopes of all the possible cases of fixed g for the assumed values of τ and γ .

IV. POWER DISSIPATED IN THE TW ACCELERATOR, RESISTIVE LOAD, AND FEEDBACK LOOP

The total power dissipated in the accelerator and in the feedback loop including the accelerator and resistive load is just $P_s(1 - \eta)$, i.e., the portion of the rf power which is not converted into beam power. The fraction of the rf power which is dissipated in the accelerator structure itself during operation at maximum efficiency can be found by setting Eq. (4) equal to Eq. (23). The result is

$$\left(\frac{P_{d(\text{accel})}}{P_s}\right)_{\eta_{\max}} = \frac{\left(\frac{\tau}{2}\right) \left(\frac{1 - e^{-\tau}}{\tau}\right)^2}{\left[(1 + g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau + \gamma)}\right]^2} \quad (26)$$

When τ and γ are small, Eq. (26) becomes:

$$\left(\frac{P_{d(\text{accel})}}{P_s}\right)_{\eta_{\max}} \approx \frac{\tau}{2g \left[\frac{1}{2g} + (\tau + \gamma) \right]^2} \quad (27)$$

Comparing Eqs. (25) and (27), it is noted that

$$\left(\frac{P_{d(\text{accel})}}{P_s}\right)_{\eta_{\max}} = (V_n)_{\eta_{\max}}^2 \quad (28)$$

Thus, the power dissipated in the accelerator is less (and the power transferred to the beam is greater) when the accelerator is designed for a high value of i_n (and a corresponding low value of V_n).

When the accelerator is designed for a maximum no-load energy, the optimizing relation [Eq. (18)] requires that $(1/2g) = \tau + \gamma$. Inserting this relation in Eq. (27) yields the result

$$\begin{array}{l} \text{(Optimum design} \\ \text{for no-load)} \end{array} \quad \frac{P_{d(\text{accel})}}{P_s} = \frac{\tau}{4(\tau + \gamma)} \quad (29)$$

Inserting the same relation in Eq. (22) gives

$$\begin{array}{l} \text{(Optimum design} \\ \text{for no-load)} \end{array} \quad \eta_{\max} = \frac{1}{1 + \frac{\tau + \gamma}{\tau + \gamma}} = 0.5 \quad (30)$$

Thus, the maximum efficiency of an accelerator designed to give maximum no-load energy is 50%. A similar procedure involving Eq. (16) gives

$$\begin{array}{l} \text{(Optimum design} \\ \text{for no-load)} \end{array} \quad P_L/P_S = 0.25 \quad (31)$$

The remaining power is lost in the feedback loop. It is given by

$$\frac{P_{\text{FB Loop}}}{P_S} = \frac{\gamma}{4(\tau + \gamma)} \quad (32)$$

The sum of the fractional powers given by Eqs. (29)-(32) is, as expected, 100%.

V. FILLING TIME OF TW ACCELERATOR WITH FEEDBACK

The power flowing in the accelerator and feedback loop may be considered as originating from two sources: (a) the rf power source; and (b) the electron beam. When the accelerator is perfectly phased, the voltages associated with these powers are in opposition and the net voltage at any point in the loop is equal to the difference of these voltages.

The beam-induced steady-state power, P_b , appearing at the output end of the accelerator can be shown⁸ to be equal to

$$P_b = i^2 r_o \ell \frac{(1 - e^{-\tau})^2}{2\tau} \quad (33)$$

Thus, the normalized power and voltage due to the beam can be written

$$\frac{P_b}{P_S} = \frac{i^2 r_o \ell}{P_S} \frac{(1 - e^{-\tau})^2}{2\tau} = i_n^2 \frac{(1 - e^{-\tau})^2}{2\tau} \quad (34)$$

and

$$\frac{V_b}{V_S} = i_n \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} \quad (35)$$

The voltage given by Eq. (35) is reduced by the factor $e^{-\gamma}$ in the feedback loop and a fraction $[g/(1+g)]^{\frac{1}{2}}$ of the resultant voltage is then sent into the accelerator arm. Thus, after one transit

$$\frac{V_o(\text{beam})}{V_S} = i_n \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} e^{-\gamma} \left(\frac{g}{1+g} \right)^{\frac{1}{2}}$$

During each successive transit through the accelerator and feedback loop the voltage

is attenuated further by the factors $e^{-\tau}$ and $e^{-\gamma}$ and the fraction $[g/(1+g)]^{\frac{1}{2}}$ is added to the existing voltage entering the accelerator. Thus, the voltage entering the accelerator builds up according to the geometrical series

$$\begin{aligned} \frac{V_{o(\text{beam})}}{V_s} &= i_n \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} e^{-\gamma} \left(\frac{g}{1+g} \right)^{\frac{1}{2}} \left\{ 1 + e^{-(\tau+\gamma)} \left(\frac{g}{1+g} \right)^{\frac{1}{2}} + \left[e^{-(\tau+\gamma)} \left(\frac{g}{1+g} \right)^{\frac{1}{2}} \right]^2 + \dots \right\} \\ &= i_n \frac{1 - e^{-\tau}}{(2\tau)^{\frac{1}{2}}} e^{-\gamma} \frac{g^{\frac{1}{2}}}{(1+g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)}} \end{aligned} \quad (36)$$

Similarly, the voltage from the power source divides with the fraction $[1/(1+g)]^{\frac{1}{2}}$ going into the accelerator arm; i.e., initially,

$$\frac{V_{o(\text{p.s.})}}{V_s} = \left(\frac{1}{1+g} \right)^{\frac{1}{2}}$$

Successive transits lead to the series

$$\begin{aligned} \frac{V_{o(\text{p.s.})}}{V_s} &= \left(\frac{1}{1+g} \right)^{\frac{1}{2}} \left\{ 1 + e^{-(\tau+\gamma)} \left(\frac{g}{1+g} \right)^{\frac{1}{2}} + \left[e^{-(\tau+\gamma)} \left(\frac{g}{1+g} \right)^{\frac{1}{2}} \right]^2 + \dots \right\} \\ &= \frac{1}{(1+g)^{\frac{1}{2}} - g^{\frac{1}{2}} e^{-(\tau+\gamma)}} \end{aligned} \quad (37)$$

Subtracting Eq. (36) from Eq. (37) gives the same result obtained earlier [Eq. (15)] using the power flow equation [Eq. (8)].

The same series is summed in obtaining both Eqs. (36) and (37). In studying the buildup of the fields in the accelerator, it is of interest to calculate the fraction of the steady-state field which is reached after n transits around the loop. Let the series be $1 + r + r^2 + \dots$, where $r = e^{-(\tau+\gamma)} [g/(1+g)]^{\frac{1}{2}}$. The sums of n terms and infinite terms are, respectively

$$S_n = \frac{1 - r^n}{1 - r} \quad (38)$$

and

$$S_\infty = \frac{1}{1 - r} \quad (39)$$

Thus,

$$\frac{S_n}{S_\infty} = 1 - r^n \quad (40)$$

The number of transits required to build up to the fraction $(1 - e^{-1})$, i.e., 63.2% of the steady-state value (S_∞) may then be calculated as follows:

$$1 - r^n = 1 - e^{-1}$$

or,

$$r^n = e^{-1}$$

or,

$$n = \frac{1}{\ln(1/r)}$$

or, using the definition of r ,

$$n = \frac{1}{(\tau + \gamma) + \frac{1}{2} \ln\left(1 + \frac{1}{g}\right)}$$

$$\approx \frac{1}{(\tau + \gamma) + \frac{1}{2g}} \quad (41)$$

Since the time for a single transit is very close to l/v_g and noting that $l/v_g = (2Q/\omega)\tau$, the time to fill the structure to the fraction $(1 - e^{-1})$ of the steady-state fields is given by $n(l/v_g)$; i.e., using Eq. (41),

$$t_F \approx \frac{(2Q/\omega)\tau}{(\tau + \gamma) + \frac{1}{2g}} \quad (42)$$

When the structure is designed for negligible beam loading, the relationship giving maximum efficiency [see Eq. (18)] is $(1/2g) = \tau + \gamma$. Thus, the filling time for this case is $(2Q/\omega)\tau g = (2Q/\omega)[\tau/(\tau + \gamma)]$. For heavier design loading, g is much smaller and $(1/2g) \gg (\tau + \gamma)$. Thus, the filling time for the heavy beam loading case approaches $(2Q/\omega)(2\tau g)$.

Using Eqs. (20) and (25), the filling time given by Eq. (42) may alternately be written

$$t_F \approx \frac{2Q}{\omega} \frac{(V_n) \eta_{\max}}{(i_n) \eta_{\max}} \quad (43)$$

Thus, the filling time is reduced as the beam current giving maximum conversion efficiency is increased (and the corresponding beam energy decreased). As an example, assuming $g = 10^5$, $\tau = 5 \times 10^{-7}$ nepers, $\gamma = 5 \times 10^{-8}$ nepers, $Q = 10^9$, and $\omega = 1.79 \times 10^{10}$ rad/sec ($f = 2856$ MHz). Then from Eqs. (20) and (25), one obtains $(i_n) \eta_{\max} = 3.17$ and $(V_n) \eta_{\max} = 0.285$. Using these values in Eq. (43), the filling time is found to be $t_F \approx 0.010$ sec.

Since, as shown previously, $V_n i_n = \eta \approx 1$ for $i_n \gg 1$, it is clear that for $i_n \gg 1$ the filling time varies as $(V_n) \eta_{\max} [\text{or, equivalently, as } (i_n) \eta_{\max}^{-2}]$.

The dependence of filling time upon frequency may be determined by recalling that $Q \propto \omega^{-2}$ and $r_0 \propto \omega^{-1}$ at superconducting temperatures.² Thus, $V_n/i_n = V/ir_0 l \propto \omega$. Then, from Eq. (43), $t_F \propto \omega^{-2}$. The rapid increase of filling time as the frequency is decreased, together with the increasing cross section of the structure, will likely be the principal factors limiting the minimum frequency.

VI. STANDING-WAVE SUPERCONDUCTING STRUCTURE WITH BEAM LOADING

For purposes of comparison, the standing-wave superconducting structure will now be examined following the results of Wilson and Schwettman.² The energy gain in this case in the presence of beam loading is given by²:

$$V_{SW} = \frac{2\beta^{\frac{1}{2}}}{1 + \beta} \left(\frac{P_s r_o \ell}{2} \right)^{\frac{1}{2}} - \frac{r_o \ell i}{2(1 + \beta)} \quad (44)$$

where β is the coupling coefficient between the transmission line and the accelerator structure. The first term on the right in Eq. (44) is the no-load energy; the second term is the reduction in no-load energy due to beam loading. Again, for convenience, Eq. (44) will be written in normalized form as follows:

$$V_{n(SW)} = \frac{(2\beta)^{\frac{1}{2}}}{1 + \beta} - \frac{i_n}{2(1 + \beta)} \quad (45)$$

where $V_n = V/(P_s r_o \ell)^{\frac{1}{2}}$ and $i_n = [(i^2 r_o \ell)/P_s]^{\frac{1}{2}}$. The conversion efficiency is then given by²:

$$\eta = \frac{V_i}{P_s} = V_n i_n = \left[\frac{(2\beta)^{\frac{1}{2}}}{1 + \beta} \right] i_n - \frac{i_n^2}{2(1 + \beta)} \quad (46)$$

For a given value of i_n , there is a value of the coupling coefficient β which results in the maximum values of V_n and η at that i_n . Maximizing V_n in Eq. (45) with respect to β (holding i_n constant) gives the optimizing relationship:

$$\beta_{(V_n, \eta)_{\max}} = 1 + \frac{i_n^2}{4} + \left[\frac{i_n^2}{4} \left(2 + \frac{i_n^2}{4} \right) \right]^{\frac{1}{2}} \quad (47)$$

Equation (47) may be used to calculate the optimum design value of β for a given value of i_n .

On the other hand, when η is maximized with respect to i_n , the necessary condition is found to be

$$(i_n)_{\eta_{\max}} = (2\beta)^{\frac{1}{2}} \quad (48)$$

When the condition of Eq. (48) is satisfied, the conversion efficiency and beam energy become²:

$$\eta_{\max} = \frac{\beta}{1 + \beta} \quad (49)$$

$$(V_n)_{\eta_{\max}} = \frac{(2\beta)^{\frac{1}{2}}}{2(1 + \beta)} \quad (50)$$

Thus; at maximum conversion efficiency, the beam energy has been reduced by beam loading to one-half of the no-load value. In general, as in the traveling-wave case, the beam energy can be written as:

$$V = V_o \left(1 - \frac{i}{2i_{\eta_{\max}}} \right) \quad (51)$$

where V_o is the no-load energy and $i_{\eta_{\max}}$ is the beam current which results in η_{\max} .

The power dissipated in the structure at maximum conversion efficiency may be found by equating Eqs. (2) and (50). The result is²:

$$\left(\frac{P_d}{P_s} \right)_{\eta_{\max}} = \frac{\beta}{(1 + \beta)^2} \quad (52)$$

Comparing Eqs. (52) and (50) it may be noted that P_d/P_s can also be written as

$$\left(\frac{P_d}{P_s} \right)_{\eta_{\max}} = 2(V_n)^2 \eta_{\max} \quad (53)$$

Thus, for the same value of normalized energy V_n , twice as much power is dissipated in the standing-wave accelerator structure as in the structure of the traveling-wave accelerator with feedback [see Eq. (28)]. From the conservation of total power, the power reflected (P_r) from the input coupler back towards the rf source may be found as follows²:

$$\frac{P_r}{P_s} = 1 - \eta_{\max} - \left(\frac{P_d}{P_s} \right)_{\eta_{\max}} = 1 - \frac{\beta}{1 + \beta} - \frac{\beta}{(1 + \beta)^2} = \frac{1}{(1 + \beta)^2} \quad (54)$$

Normalized values of beam energy (V_n), beam current (i_n), beam conversion efficiency (η), power dissipated in accelerator structure (P_d/P_s), and reflected power (P_r/P_s) vs coupling coefficient β , all at maximum conversion efficiency, are shown in Fig. 4. The maximum efficiency obtainable improves as β increases. Other advantages of high β are the smaller fraction of the source power reflected and the smaller fraction of the power dissipated in the accelerator structure. A disadvantage of increasing β is the decreasing magnitude of V_n at η_{\max} .

VII. FILLING TIME OF STANDING-WAVE STRUCTURE

For $\beta \gg 1$, most of the power is transferred to the beam and the power dissipated in the structure, given by Eq. (52), becomes $P_d/P_s \approx 1/\beta$. Thus, the loaded Q, Q_L , becomes

$$Q_L \approx \frac{Q}{\beta} \approx Q \frac{P_d}{P_s} \approx \frac{Q P_d}{i \eta_{\max} V \eta_{\max}} \quad (55)$$

Combining Eqs. (55) and (2) yields the result²

$$Q_L \approx \frac{2V \eta_{\max} / \ell}{i \eta_{\max} (r_o / Q)} = 2Q \frac{(V_n) \eta_{\max}}{(i_n) \eta_{\max}} \quad (56)$$

The time to fill the structure to $1 - 1/e$, i.e., to 63.2% of the magnitude of the steady-state field, is $2Q_L/\omega$. Thus, the time to fill to this level is:

$$t_F \approx \frac{4Q}{\omega} \frac{(V_n) \eta_{\max}}{(i_n) \eta_{\max}} \quad (57)$$

Comparing Eqs. (43) and (57) one notes that, for equal beam energies, currents, lengths, and r_o/Q , the filling time of the standing-wave structure is twice that of the traveling-wave structure with feedback. The basic reason for this result is that the standing-wave accelerator is filled by successive reflections in the accelerator structure itself,

whereas the traveling-wave structure is filled by successive feedback of the power through the external loop in which the transit time is negligible compared to the one-way transit time in the accelerator structure.

As in the traveling-wave case, for $i_n \gg 1$, the filling time varies as $(V_n) \eta_{\max}^2$ or, equivalently, as $(i_n) \eta_{\max}^{-2}$. Also, with other parameters fixed, $t_F \propto \omega^{-2}$.

VIII. COMPARISON OF TRAVELING-WAVE AND STANDING-WAVE DESIGNS

The normalized beam energy V_n and beam conversion efficiency η in the standing-wave accelerator are plotted in Fig. 5 vs i_n for three values of the coupling coefficient β ($\beta = 1, 10, \text{ and } 50$). Also shown in this figure with dashed lines are the envelopes of the V_n and η families representing the maximum possible values of these variables at each value of i_n .

The envelopes of the entire family of g values for the traveling-wave accelerator with feedback and the β values of the standing-wave accelerator are plotted together in Fig. 6. As noted above, these curves represent the maximum possible values of V_n and η at each value of i_n and hence are useful in the theoretical comparison of the various possible designs. It is noted that the traveling-wave accelerator with feedback excels with respect to the maximum beam energy obtainable. At light loading it also has a higher conversion efficiency than the standing-wave accelerator. With increasing beam loading, the energies and efficiencies of both of these basic types approach equality.

The advantages relating to higher energy and higher efficiency stem solely from the reduced loss in the feedback loop compared to the "internal" feedback through the accelerator structure in the standing-wave accelerator. If $\tau = \gamma$, i.e., if the losses in the feedback loop are equal to the internal losses, the two accelerator types have the same maximum values of V_n and η at all values of i_n . However, the advantage of reduced ratio of peak to average fields for the traveling-wave accelerator with feedback still remains even if $\tau = \gamma$.

The practical realization of a superconducting accelerator with feedback may turn out to be quite complicated due to the requirement to supercool the external feedback loop. Also, careful attention must be given to the elimination of rf reflections in the loop in order to prevent the build-up of a backward wave of significant amplitude. This backward wave builds up at the expense of the forward wave and hence would result in a reduction of beam energy and conversion efficiency. A tuner in the feedback loop might be needed to compensate for residual reflections. Ideally, this tuner would be automatically controlled from a signal derived from the backward wave.

In summary, the superconducting traveling-wave accelerator with feedback has the theoretical advantages of somewhat higher energies and efficiencies at light loading as noted above and also the advantage of reduced ratio of peak to average fields in the accelerator cavities. The latter characteristic may be a definite advantage if the maximum energy gradient obtainable is limited by either the critical magnetic field for the superconductor or by field emission. Moreover, for the same values of normalized energies V_n , the traveling-wave accelerator with feedback has one-half as much rf power dissipated in the accelerator structure as the standing-wave accelerator. Because of the high cost of providing refrigeration at superconducting temperatures, this is an important consideration. In addition, for equal beam energies, beam currents, and r_0/Q , the traveling-wave accelerator with feedback has one-half the filling time of the standing-wave accelerator. This characteristic may become important if unity duty cycle is not feasible, or if it becomes desirable to turn off the rf power periodically, e.g., while the beam is being switched from one research area to another.

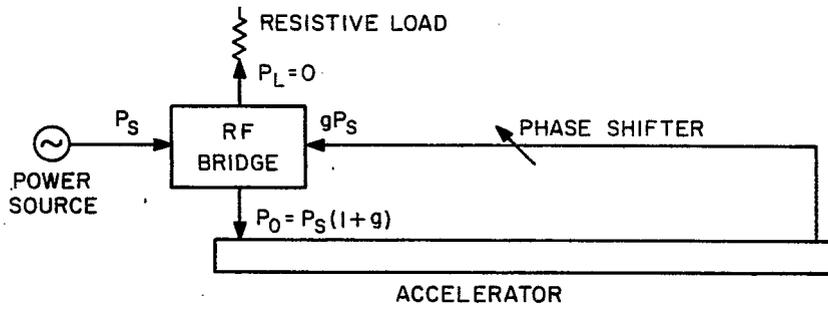


Fig. 1. Schematic diagram illustrating feedback principle. Relations shown represent steady-state conditions with correct bridge ratio.

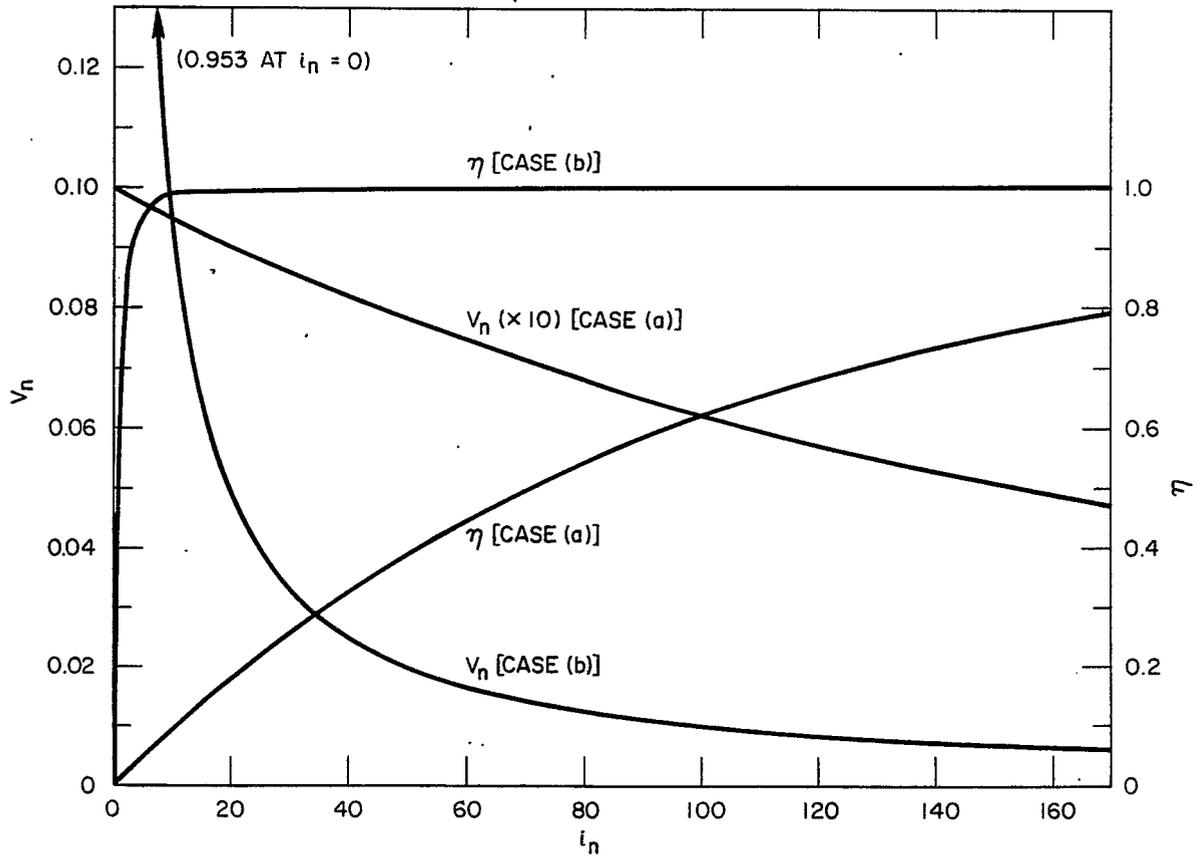


Fig. 2. Normalized beam energy (V_n) and beam conversion efficiency (η) for traveling-wave superconducting accelerator with optimum feedback vs normalized beam current (i_n). Accelerator attenuation parameter, $\tau = 5 \times 10^{-7}$ nepers. Feedback loop attenuation parameter, $\gamma = 5 \times 10^{-3}$ nepers [Case (a)] and $\gamma = 5 \times 10^{-8}$ nepers [Case (b)].

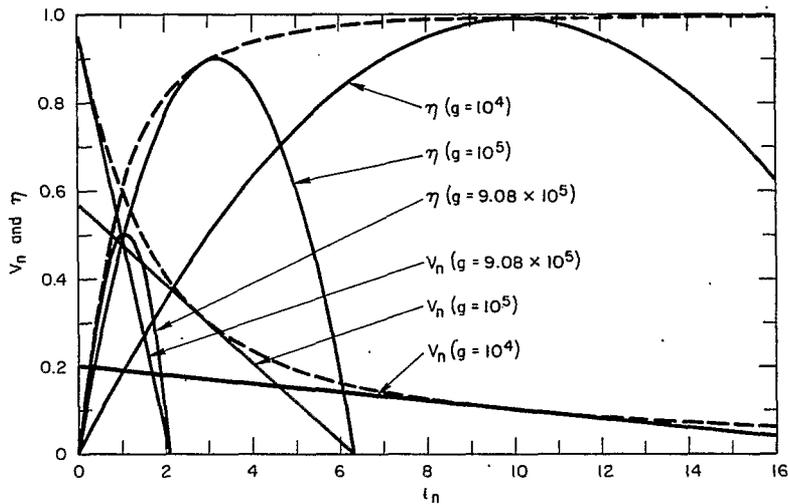


Fig. 3. Normalized beam energy (V_n) and beam conversion efficiency (η) for traveling-wave superconducting accelerator with feedback vs normalized beam current (i_n). Curves are shown for three fixed bridge ratios. Accelerator attenuation parameter, $\tau = 5 \times 10^{-7}$ nepers. Feedback loop attenuation parameter, $\gamma = 5 \times 10^{-8}$ nepers.

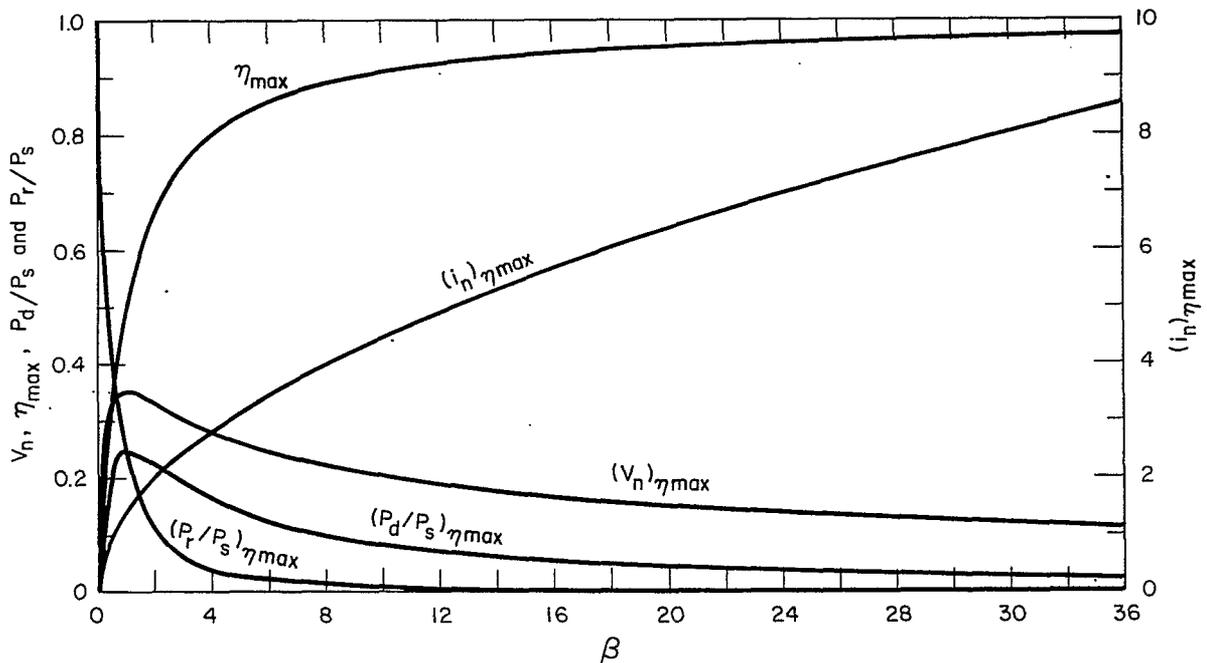


Fig. 4. Normalized values of beam energy (V_n), beam current (i_n), beam conversion efficiency (η), power dissipated in accelerator structure (P_d/P_s), and reflected power (P_r/P_s) in superconducting standing-wave accelerator vs coupling coefficient β , all at maximum beam conversion efficiency.

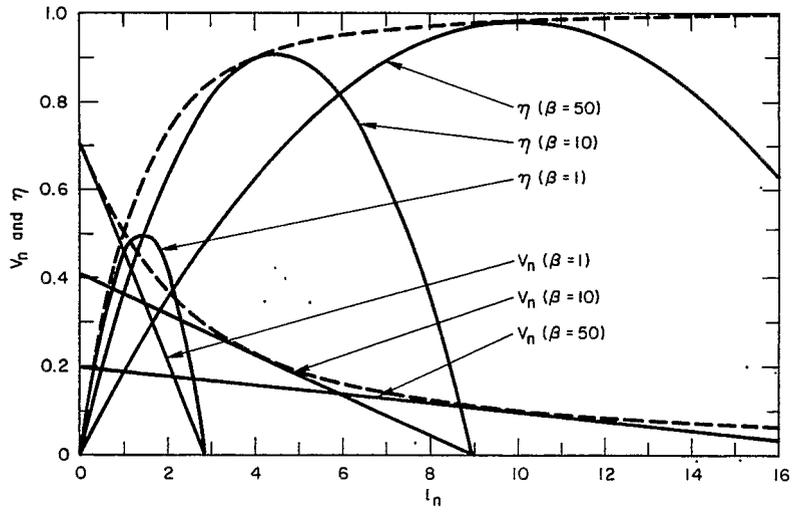


Fig. 5. Normalized beam energy (V_n) and beam conversion efficiency (η) for standing-wave superconducting accelerator vs normalized beam current (i_n). Curves are shown for three fixed values of the coupling coefficient β .

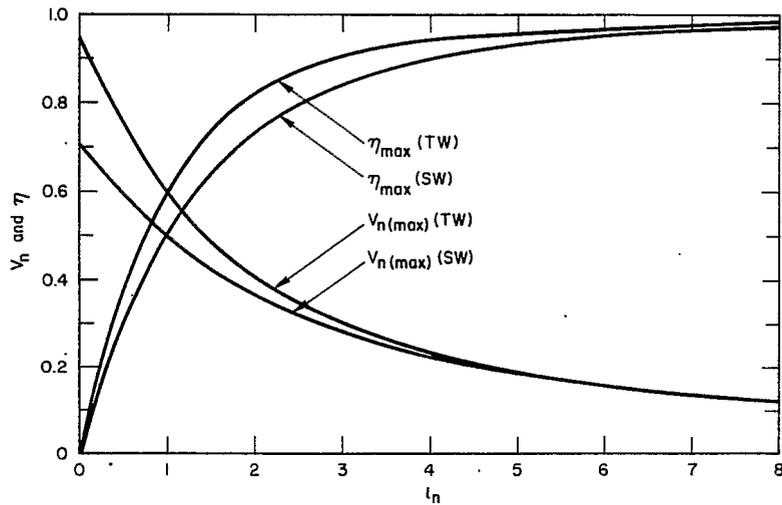


Fig. 6. Comparison of maximum values of normalized beam energy (V_n) and beam conversion efficiency (η) vs normalized beam current (i_n) for superconducting traveling-wave and standing-wave accelerators. For traveling-wave accelerator, accelerator attenuation parameter $\tau = 5 \times 10^{-7}$ nepers and feedback loop attenuation parameter $\gamma = 5 \times 10^{-8}$ nepers.